Brill–Noether Special K3 Surfaces

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Abstract

We study the lifting of linear systems on curves in polarized K3 surfaces and prove a bounded version of the Donagi–Morrison conjecture for rank 3 linear systems. Using these developments, and a study of Lazarsfeld–Mukai bundles, we prove that a polarized K3 surface of genus $g \leq 17$ is Brill–Noether special if and only if a curve in the polarization class is Brill–Noether special.

Brill–Noether Theory (Curves)

Let C be a curve and $A \in \mathrm{Pic}(C)$ a line bundle. We say A is a g^r_d when $h^0(C, A) = r+1$ and $\deg(A) = d$. The *Clifford index* of A is $\gamma(A) = d - 2r$. The Clifford index of C is $\gamma(C) := \min\{\gamma(A) | A \in \mathrm{Pic}(C), h^0(C, A), h^1(C, A) \geq 2\}$.

The Brill–Noether theorem states that when

$$\rho(g, r, d) = \underbrace{g}_{\text{genus}} - \underbrace{(r+1)}_{h^0(C,A)} \underbrace{(g - d + r)}_{h^0(C, \omega_C - A)} \geq 0,$$

then C admits a g^r_d. Therefore $\gamma(C) \leq \lfloor \frac{g-1}{2} \rfloor$.

Moreover, if $\rho(g, r, d) < 0$ then a general curve of genus g has no g^r_d. A line bundle A with $\rho(A) < 0$ is called *Brill–Noether special*, and a curve admitting such a line bundle is also called *Brill–Noether special*.

Brill–Noether Theory (K3 surfaces)

Let (S, H) be a polarized K3 surface of genus g (degree $2g-2$). That is, $H^2 = 2g-2$, and a smooth curve $C \in |H|$ has genus g.

Definition: [Mukai] (S, H) is *Brill–Noether special* if there is a nontrivial $J \neq H \in \mathrm{Pic}(S)$ such that

$$g - h^0(S, J) h^0(S, H - J) < 0.$$

Else (S, H) is called *Brill–Noether general*.

Proposition: If (S, H) is Brill–Noether special, then C is Brill–Noether special.

Theorem [4]: If $\mathrm{Pic}(S) = \mathbb{Z}H$, then $C \in |H|$ is Brill–Noether general.

So if C is Brill–Noether special, then $\mathrm{rk}\,\mathrm{Pic}(S) \geq 2$.

In particular, $\mathrm{Pic}(S)$ admits a primitive embedding of the lattice

$$\Lambda^r_{g,d} = \begin{array}{c|cc} & H & L \\ \hline H & 2g-2 & d \\ L & d & 2r-2 \end{array}.$$

In the moduli space $\mathcal{K}_g$ of polarized K3 surfaces of genus g, there is a Noether–Lefschetz divisor $\mathcal{K}^r_{g,d}$ parameterizing such polarized K3 surfaces.

Conjecture and Theorem

Brill–Noether special K3 conjecture: Let (S, H) be a polarized K3 surface of genus $g \geq 2$. Then (S, H) is Brill–Noether special if and only if a curve $C \in |H|$ is Brill–Noether special.

Strategy: Suppose that C admits a Brill–Noether special line bundle A. Then find a Donagi–Morrison lift $M \in \mathrm{Pic}(S)$ of A and use M to find the required line bundle J making (S, H) Brill–Noether special.

Theorem [Auel–H.]: The conjecture holds in genus $2 \leq g \leq 17$

In genus ≥ 17, similar techniques can prove the conjecture, however, additional results regarding lifts of Brill–Noether special line bundles are needed.

Lattice Restrictions

For a polarized K3 surface with $\mathrm{Pic}(S) = \Lambda^r_{g,d}$ to exist, the Hodge index theorem implies

$$\Delta(g, r, d) := \mathrm{disc}(\Lambda^r_{g,d}) = 4(r-1)(g-1) - d^2 < 0.$$

Proposition [3]: The locus of Brill–Noether special K3 surfaces in $\mathcal{K}_g$ is a union of the Noether–Lefschetz divisors $\mathcal{K}^r_{g,d}$ satisfying $2 \leq d \leq g-1$, $\Delta(g, r, d) < 0$, and $\rho(g, r, d) < 0$.

Lifting Brill–Noether Special Line Bundles

Let $A \in \mathrm{Pic}(C)$ be a Brill–Noether special line bundle. We are interested in finding a lift of A to a line bundle $M \in \mathrm{Pic}(S)$. We do this by studying the lifting of line bundles on polarized K3 surfaces.

Donagi–Morrison Conjecture [1, 6]: Let (S, H) be a polarized K3 surface and $C \in |H|$ be a smooth irreducible curve of genus ≥ 2. Suppose A is a complete basepoint free g^r_d on C such that $d \leq g-1$ and $\rho(g, r, d) < 0$. Then there exists a line bundle $M \in \mathrm{Pic}(S)$ adapted to $|H|$ such that

- $|A|$ is contained in the restriction of $|M|$ to C, and
- $\gamma(M \otimes \mathcal{O}_C) \leq \gamma(A)$.

The line bundle M is called a *Donagi–Morrison lift of* A.

Donagi and Morrison verified the Donagi–Morrison conjecture for $r = 1$, and Lelli-Chiesa verified it for $r = 2$ [1, 5] and when $\gamma(A) = \gamma(C)$ [6]. These lifting results prove the Brill–Noether special K3 conjecture when $\gamma(A) \leq \gamma(C)$.

Genus ≥ 14

In genus $g \geq 14$, there are Brill–Noether special line bundles with $\gamma(A) > \gamma(C)$.

In genus 14, a general curve has Clifford index $\gamma(C) = 6$, however there are two Brill–Noether line bundles with $\gamma = 7$: g^2_{11} and g^3_{13}.

Lifting g^3_ds

Theorem [2]: Let (S, H) be a polarized K3 surface of genus $g \neq 2, 3, 4, 8$, and $C \in |H|$ a smooth irreducible curve of Clifford index $\gamma(C)$. Then there is a constant $\kappa(\gamma(C), \mathrm{Pic}(S))$ such that if $d < \kappa$ then the Donagi–Morrison conjecture holds for any g^3_d on C.

Proof Idea

Not every Donagi–Morrison lift M makes (S, H) Brill–Noether special!!

Find new line bundle $K \in \mathrm{Pic}(S)$.

$$\begin{array}{c|ccc} & H & M & K \\ \hline H & 2g-2 & e & K.H \\ M & e & 2s-2 & K.M \\ K & K.H & K.M & K^2 \end{array} \subseteq \mathrm{Pic}(S).$$

Maybe some combination of H, M, and K will work!

Lazarsfeld–Mukai Bundles

We define a bundle $F_{C,A}$ on S via the short exact sequence

$$0 \longrightarrow F_{C,A} \longrightarrow H^0(C, A) \otimes \mathcal{O}_S \xrightarrow{ev} \iota_*(A) \longrightarrow 0.$$

Dualizing gives $E_{C,A} = F^\vee_{C,A}$ (the LM bundle associated to A on C) sitting in the short exact sequence

$$0 \longrightarrow H^0(C, A)^\vee \otimes \mathcal{O}_S \longrightarrow E_{C,A} \longrightarrow \iota_*(\omega_C \otimes A^\vee) \longrightarrow 0;$$

The LM bundle $E_{C,A}$ is like a lift of A to a vector bundle on S.

Let $E_{C,A}$ be a LM bundle associated to a basepoint free line bundle A of type g^r_d on $C \subset S$, then:

- $\mathrm{rk} = r+1$, $c_1 = H = [C]$, $c_2 = d$
- $E_{C,A}$ is globally generated off the base locus of $\iota_*(\omega_C \otimes A^\vee)$
- If $\rho(A) < 0$, then $E_{C,A}$ is not stable

Proposition: Suppose $N \in \mathrm{Pic}(S)$ is a globally generated line bundle and

$$0 \to N \to E_{C,A} \to E \to 0$$

is exact, with E stable. Then $M := \det E$ is a Donagi–Morrison lift of A.

Generalized Lazarsfeld–Mukai Bundles

Definition: A *generalized Lazarsfeld–Mukai bundle* is a torsion free coherent sheaf E such that $h^2(S, E) = 0$ and either

(I) E is locally free and globally generated off finitely many points; or
(II) E is globally generated.

The Clifford index of E is $\gamma(E) := c_2(E) - 2(\mathrm{rk}(E) - 1)$.

Proposition: When A and $\omega_C \otimes A^\vee$ are basepoint free, the quotient $E := E_{C,A}/N$ is a generalized Lazarsfeld–Mukai bundle of type (II).

- $\gamma(E_{C,A}) = d - 2r = \gamma(A)$
- $\gamma(E) = \gamma(A) - M.H + M^2 + 2$ "$=$" $\gamma(A) - \gamma(M|_C)$

Genus ≤ 17: Can assume $0 \leq \gamma(E) \leq 2$, and $E = E_{D,B}$ for a smooth irreducible curve D and line bundle B. Lift B to $K \in \mathrm{Pic}(S)$. Taking $J = M$ or $J = M - K$ works!

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References

[1] Ron Donagi and David R. Morrison, *Linear systems on K3-sections*, J. Differential Geom. **29** (1989), no. 1, 49–64.

[2] Asher Auel and Richard Haburcak, *Maximal Brill–Noether loci via K3 surfaces*, 2022.

[3] Francois Greer, Zhiyuan Li, and Zhiyu Tian, *Picard Groups on Moduli of K3 Surfaces with Mukai Models*, International Mathematics Research Notices **2015** (2014), no. 16, 7238–7257.

[4] Robert Lazarsfeld, *Brill–Noether-Petri without degenerations*, Journal of Differential Geometry **23** (1986), no. 3, 299 – 307.

[5] Margherita Lelli-Chiesa, *Stability of rank-3 Lazarsfeld-Mukai bundles on K3 surfaces*, Proc. Lon. Math. Soc. **107** (2013), no. 2, 451–479.

[6] Margherita Lelli-Chiesa, *Generalized Lazarsfeld-Mukai bundles and a conjecture of Donagi and Morrison*, Adv. Math. **268** (2015), no. 2, 529–563.

[7] Margherita Lelli-Chiesa, *A codimension 2 component of the Gieseker–Petri locus*, 2021, to appear J. Algebraic Geom.