Brill–Noether special K3 surfaces

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Brill–Noether Theory (Curves)

Classical Brill–Noether theory studies linear systems on curves. We'll think of these as line bundles A on a curve C.

Call a line bundle A of *type* g^r_d if $h^0(C, A) = r + 1$ and $\deg(A) = d$.
In other words, $|A|$ defines a map $C \to \mathbb{P}^r$ of degree d.

Definition

The *Brill–Noether number* is

$$\rho(g, r, d) = \underbrace{g}_{\text{genus}} - \underbrace{(r+1)}_{h^0(C,A)}\underbrace{(g-d+r)}_{h^0(C,\omega_C - A)}.$$

Brill–Noether Theory (Curves)

Theorem (Brill–Noether Theorem)

$$\dim\{g^r_d \textit{ on } C\} \geq \rho(g, r, d).$$

When C is ***general****,*

$$\dim\{g^r_d \textit{ on } C\} = \rho(g, r, d).$$

Thus when $\rho(g, r, d) < 0$, a general curve has **no** g^r_d.

Definition

A line bundle A with $\rho(A) < 0$ is called *Brill–Noether special*. A curve admitting a Brill–Noether special line bundle A is called *Brill–Noether special*. Otherwise, C is called *Brill–Noether general*.

Brill–Noether Theory (Curves)

Example: genus 2

Every genus 2 curve is hyperelliptic (has a g^1_2):

$$\rho(2,1,2) = 2 - (2 - 2 + 1) = 1.$$

Example: genus 3

Not every genus 3 curve is hyperelliptic (has a g^1_2):

$$\rho(3,1,2) = 3 - (2)(3 - 2 + 1) = -1.$$

Brill–Noether Theory (K3s)

Let (S, H) be a polarized K3 surface of genus g (degree $2g-2$).
That is, $H^2 = 2g-2$, and a smooth curve $C \in |H|$ has genus g.

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Definition (Mukai)

(S, H) is *Brill–Noether special* if there is a nontrivial $J \neq H \in \mathrm{Pic}(S)$ such that

$$g - h^0(S, J)h^0(S, H - J) < 0.$$

Else (S, H) is called *Brill–Noether general.*

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Theorem (Lazarsfeld)

If $\operatorname{Pic}(S) = \mathbb{Z}H$, *then* $C \in |H|$ *is Brill–Noether general.*

So if C is Brill–Noether special, then $\operatorname{rk}\operatorname{Pic}(S) \geq 2$.

Brill–Noether Theory (K3s)

What do the Picard groups of Brill–Noether special K3s look like?

Lattices

Let $\mathcal{K}_g$ be the moduli space of primitively quasi-polarized K3 surfaces of genus g.
The Noether–Lefschetz divisor $\mathcal{K}^r_{g,d} \subset \mathcal{K}_g$ parameterizes K3 surfaces with a specific lattice polarization

$$\Lambda^r_{g,d} := \begin{array}{c|cc} & H & L \\ \hline H & 2g-2 & d \\ L & d & 2r-2 \end{array} \subseteq \mathrm{Pic}(S).$$

$$\Delta(g,r,d) := \mathrm{disc}(\Lambda^r_{g,d}) = 4(r-1)(g-1) - d^2$$

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Proposition (Greer–Li–Tian)

The locus of Brill–Noether special K3 surfaces in $\mathcal{K}_g$ is a union of the Noether–Lefschetz divisors $\mathcal{K}^r_{g,d}$ satisfying $2 \leq d \leq g-1$, $\Delta(g,r,d) < 0$, and $\rho(g,r,d) < 0$.

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Proof.

Restrict J to C. $\rho(J|_C) = g - h^0(C, J|_C)h^0(C, \omega_C - J|_C)$.
(Recall $\omega_C = H|_C$ by adjunction) □

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Question (Knutsen, Mukai)

Is the converse true?

Conjecture

Let (S, H) be a polarized K3 surface of genus g. Then (S, H) is Brill–Noether special if and only if a curve $C \in |H|$ is Brill–Noether special.

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Theorem (Mukai, Knutsen)

The conjecture holds in genus $g \leq 10$, and $g = 12$.

Proved using Mukai models of K3s.

Brill–Noether Theory (K3s)

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Theorem (Auel–H.)

The conjecture holds in genus $g \leq 17$.

Lifting line bundles

Conjecture

Let (S, H) be a polarized K3 surface. Then (S, H) is Brill–Noether special if and only if a curve $C \in |H|$ is Brill–Noether special.

Theorem (Auel–H.)

Conjecture holds in genus $g \leq 17$.

Idea

If C is Brill–Noether special, say it has some line bundle A with $\rho(A) < 0$, can we *lift* A to a line bundle $L \in \mathrm{Pic}(S)$ so that L makes (S, H) Brill–Noether special?

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So can we always lift a Brill–Noether special line bundle (to make S Brill–Noether special)?

NO!

First counterexample by Donagi–Morrison (1989) which disproved a (different) conjecture by Harris and Mumford on the constancy of the gonality of curves on K3 surfaces.

What line bundles can we lift?

Lifting Line Bundles

Let $C \in |H|$ be a smooth irreducible curve of genus $g \geq 2$.

Theorem

- *(Saint-Donat) Let A be a g^1_2 on C, then it lifts.*
- *(Reid) Let A be a g^1_d on C, then if $d < \kappa(g)$, it lifts.*

Lifting Line Bundles

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- *(Reid) Let* A *be a* g^1_d *on* C*, then if* $d < \kappa(g)$*, it lifts.*

Theorem (Donagi–Morrison)

Let A *be a* g^1_d *with* $\rho(A) < 0$*. Then there is a line bundle* $M \in \mathrm{Pic}(S)$ *such that*

- M *is adapted to* H*,*
- $|A| \subseteq |M||_C$*, and*
- $\gamma(M|_C) \leq \gamma(A)$*.*

(Constrains M^2 *and* $H.M$*)*

Call M a Donagi–Morrison lift of A.

Clifford Index Interlude

The *Clifford index of a line bundle* A of type g^r_d on C is

$$\gamma(A) := d - 2r.$$

The *Clifford index of a curve* C is

$$\gamma(C) := \min\left\{\gamma(A) \mid h^0(C,A), h^0(C,\omega_C - A) \geq 2\right\}.$$

Fact

$$\underbrace{0 \leq}_{\text{Clifford}} \gamma(C) \underbrace{\leq \left\lfloor \frac{g-1}{2} \right\rfloor}_{\text{BN Theory}}$$

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Fact

$$\underbrace{0 \leq}_{\text{Clifford}} \gamma(C) \underbrace{\leq \left\lfloor \frac{g-1}{2} \right\rfloor}_{\text{BN Theory}}$$

- $\gamma(C) = 0 \iff C$ is hyperelliptic.
- $\gamma(C) = 1 \iff C$ has a g^1_3 or a g^2_5.

Donagi–Morrison Conjecture

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Suppose A is a complete basepoint free g^r_d on C with $d \leq g - 1$ and $\rho(A) < 0$.
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Theorem

- *(Donagi–Morrison) Conjecture holds for* $r = 1$.
- *(Lelli-Chiesa) Conjecture holds for* $r = 2$.
- *(Lelli-Chiesa) Conjecture holds if* $\gamma(A) = \gamma(C)$, *except for finitely many explicit cases.*

Conjecture

Let (S, H) be a polarized K3 surface. Then (S, H) is Brill–Noether special if and only if a curve $C \in |H|$ is Brill–Noether special.

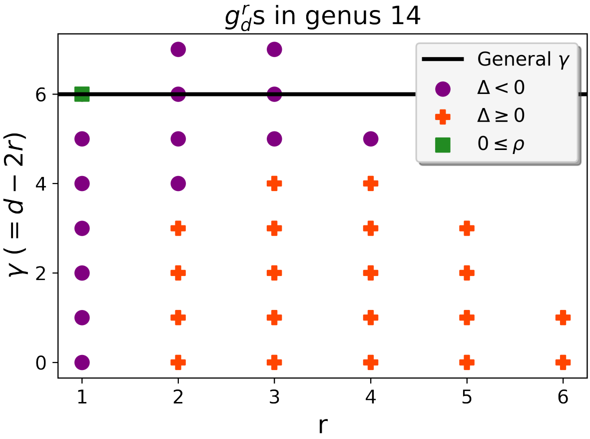
Theorem (Auel–H.)

Conjecture holds in genus ≤ 17.

Slogan: Lifting results $\implies$ *Conjecture*
Are the previous lifting results enough?

Non-computing g^r_ds

In genus $g \geq 14$, there are Brill–Noether special line bundles A with $\gamma(A) > \left\lfloor \frac{g-1}{2} \right\rfloor \geq \gamma(C)$.



So we need more lifting results.

Rank 3 Donagi–Morrison Conjecture & Genus 14

Theorem (Auel–H.)

Let A be a complete basepoint free g^3_d with $d \leq g-1$ on $C \subset S$ with $\rho(A) < 0$ and $d < \kappa(\gamma(C), \mathrm{Pic}(S))$.
Then there is a Donagi–Morrison lift M of A.

Maximal Brill–Noether loci via K3 surfaces (Auel–H., 2022)
arxiv: 2206.04610

Higher Dimensional Brill–Noether Theory?

- Classical: Curves ($\rho < 0$)
- Mukai: K3 surfaces ($J \in \mathrm{Pic}(S)$ with "$\rho < 0$")
- Mukai: Fano 3-folds (anti-canonical section is Brill–Noether special)
- Auel: Cubic 4-folds (Hodge associated Brill–Noether special K3)

Question

What is a good notion of "Brill–Noether special" for hyperkähler varieties?

Genus 14

Theorem (Auel–H.)

Let (S, H) be a polarized K3 of genus 14. Then (S, H) is Brill–Noether special if and only if $C \in |H|$ is Brill–Noether special.

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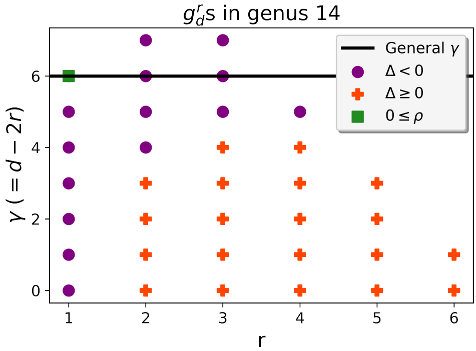
We argue by the Clifford index of C. Suppose C is Brill–Noether special, having a line bundle A with Clifford index $\gamma(A)$.



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- If $\gamma(C) < 6$, then $\gamma(A) < 6$.
  - We apply Lelli-Chiesa's lifting results. The Donagi–Morrison lift M makes (S, H) Brill–Noether special.

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- If $\gamma(C) = 6$, and $\gamma(A) = 7$:
  - we apply the known $r = 2$ or $r = 3$ cases of the Donagi–Morrison conjecture.

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- If $\gamma(C) = 6$, and $\gamma(A) = 7$:
  - we apply the known $r = 2$ or $r = 3$ cases of the Donagi–Morrison conjecture.
  - Sometimes the lift M does not make (S, H) Brill–Noether special!

$$\text{Have} \quad \begin{array}{c|cc} & H & M \\ \hline H & 26 & e \\ M & e & 2s-2 \end{array} \subseteq \mathrm{Pic}(S).$$

But M does not always make (S, H) Brill–Noether special.

So need another line bundle!

Obtain new line bundles from the construction of Donagi–Morrison lifts.

Using...

Lazarsfeld–Mukai Bundles

Lazarsfeld–Mukai Bundles

Let A be a basepoint free complete g^r_d on $\iota : C \hookrightarrow S$.

Definition

There is an exact sequence

$$0 \to F_{C,A} \to H^0(C, A) \otimes \mathcal{O}_S \to \iota_* A \to 0.$$

Dualizing gives

$$0 \to H^0(C, A)^\vee \otimes \mathcal{O}_S \to E_{C,A} \to \iota_*(\omega_C \otimes A^\vee) \to 0.$$

The vector bundle $E_{C,A}$ is the *Lazarsfeld–Mukai bundle associated to* A.

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Properties of $E_{C,A}$

- $\mathrm{rk} = r + 1$, $c_1 = H = [C]$, $c_2 = d$
- $E_{C,A}$ is globally generated off the base locus of $\iota_*(\omega_C \otimes A^\vee)$
- If $\rho(A) < 0$, then $E_{C,A}$ is not stable

LM bundles and Lifting

Proposition

Suppose $N \in \mathrm{Pic}(S)$ *is a globally generated line bundle and*

$$0 \to N \to E_{C,A} \to E \to 0$$

is exact, with E *stable. Then* $M := \det E$ *is a Donagi–Morrison lift of* A*.*

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Generalized LM bundles (gLM)

A *generalized Lazarsfeld–Mukai bundle* is a torsion free coherent sheaf E such that $h^2(S, E) = 0$ and either

1. E is locally free and globally generated off finitely many points; or
2. E is globally generated.

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Definition

Let E be a gLM bundle.
The Clifford index of E is $\gamma(E) := c_2(E) - 2(\text{rk}(E) - 1)$.

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$$\gamma(E) = \gamma(A) - \gamma(M|_C)$$

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- Thus D has a g^1_2, a g^1_3, or a g^2_5.

These lift to a line bundle $K \in \mathrm{Pic}(S)$!

Have
$$\begin{array}{c|ccc} & H & M & K \\ \hline H & 2g-2 & e & d \\ M & e & 2s-2 & \{2,3,5\} \\ K & d & \{2,3,5\} & \{0,0,2\} \end{array} \subseteq \mathrm{Pic}(S).$$

Taking $J = K$ or $J = M - K$ shows that (S, H) is Brill–Noether special!

Brill–Noether Special K3s

Theorem (Auel–H.)

Let (S, H) be a polarized K3 surface of genus $g \leq 17$. Then S is Brill–Noether special if and only if a smooth irreducible curve $C \in |H|$ is Brill–Noether special.

$$\begin{array}{c}\text{Known Lifting Results} \\ + \\ \text{LM Bundles}\end{array} \Longrightarrow \text{Brill–Noether Special K3s}$$

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Donagi–Morrison Conjecture

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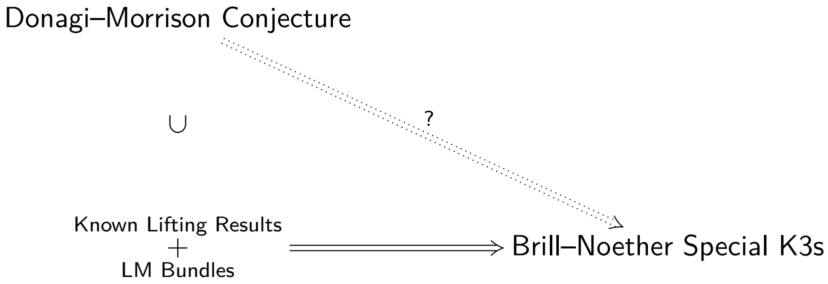

Questions?

Proving Donagi–Morrison in Rank 3

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Let A be a complete basepoint free g^3_d with $d \leq g-1$ on C with $\rho(A) < 0$ and $d < \kappa(\gamma(C), \mathrm{Pic}(S))$.
Then there is a Donagi–Morrison lift M of A.

The Lazarsfeld–Mukai bundle $E_{C,A}$ has $\operatorname{rk} E_{C,A} = 4$, and is unstable.

Proving Donagi–Morrison in Rank 3

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Expanding the Harder–Narasimhan and Jordan–Hölder filtrations of $E_{C,A}$, we obtain a *terminal filtration*

$$0 \subset E_1 \subset \cdots \subset E_4 = E_{C,A}$$

where the quotients are stable.

Proving Donagi–Morrison in Rank 3

Proposition

Suppose $N \in \mathrm{Pic}(S)$ is a globally generated line bundle and

$$0 \to N \to E_{C,A} \to E \to 0$$

is exact, with E stable. Then $M := \det E$ is a Donagi–Morrison lift of A.

Want the terminal filtration of $E_{C,A}$ to look like

$$0 \subset N \subset E_{C,A} \qquad (\text{type } 1 \subset 4)$$

for a line bundle N.

"Theorem"

If the terminal filtration of $E_{C,A}$ is not of type $1 \subset 4$, then $c_2(E_{C,A}) = d \gg 0$.

Proving Donagi–Morrison in Rank 3

Idea

Depending on the filtration type,

$$c_2(E_{C,A}) = c_2 \text{ terms } + c_1.c_1' \text{ terms} \geq \kappa$$

We bound the c_2 terms using the dimension of stable sheaves with given Mukai vector, and the products of c_1 terms using slope arguments.

Thus when $c_2(E_{C,A}) = d < \kappa(\gamma(C), \mathrm{Pic}(S))$, we only have a filtration of type $1 \subset 4$, and we have a Donagi–Morrison lift!

Thank You!

Questions?