Maximal Brill–Noether loci

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Classical Brill–Noether theory

Let C be a smooth algebraic curve. Brill–Noether theory studies the maps $C \to \mathbb{P}^r$.

Such a map is given by a g^r_d:
a pair (A, V) of

- a line bundle $A \in \mathrm{Pic}^d(C)$ with $h^0(C, A) \geq r + 1$, and
- a subspace $V \subseteq H^0(C, A)$ of dimension $r + 1$.

Question (Brill–Noether Theory)

What g^r_d's does C have?

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Brill–Noether loci

Theorem (Gieseker, Griffiths–Harris, Lazarsfeld)

A general curve $C \in \mathcal{M}_g$ admits a g^r_d if and only if

$$\rho(g, r, d) := g - (r+1)(g - d + r) \geq 0$$

Thus when $\rho(g, r, d) < 0$, the *Brill–Noether locus*

$$\mathcal{M}^r_{g,d} := \{C \in \mathcal{M}_g \text{ admitting a } g^r_d\} \text{ is a subvariety of } \mathcal{M}_g.$$

Question (Refined Brill–Noether theory)

For a "general" curve in $\mathcal{M}^r_{g,d}$, what g^s_e's does it have?

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What g^s_e's does a "general" curve in $\mathcal{M}^r_{g,d}$ admit?

When $r = 1$, $\mathcal{M}^1_{g,d}$ is irreducible. For $r \geq 2$, $\mathcal{M}^r_{g,d}$ can have multiple components!

We can get a coarse picture of a refined Brill–Noether theory by understanding the relative positions of Brill–Noether loci.

We have trivial containments

- $\mathcal{M}^r_{g,d} \subset \mathcal{M}^r_{g,d+1}$ by adding a basepoint
- $\mathcal{M}^r_{g,d} \subset \mathcal{M}^{r-1}_{g,d-1}$ by subtracting a non-basepoint

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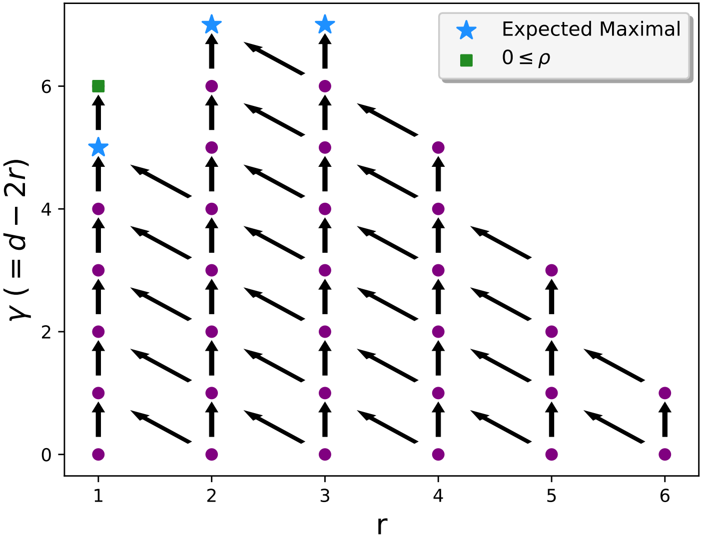
Brill–Noether loci in genus 14



Maximal Brill–Noether loci

In genus $7, 8, 9$, there are non-trivial containments:
$\mathcal{M}^2_{7,6} \subset \mathcal{M}^1_{7,4}$, $\mathcal{M}^1_{8,4} \subset \mathcal{M}^2_{8,7}$, $\mathcal{M}^2_{9,7} \subset \mathcal{M}^1_{9,5}$. [Larson, Mukai]

Conjecture (Auel–H.)

For $g \geq 3$, except $g = 7, 8, 9$, the expected maximal Brill–Noether loci are maximal.

That is, for every pair of expected maximal loci there is some curve $C \in \mathcal{M}^r_{g,d}$ but $C \notin \mathcal{M}^s_{g,e}$.

The conjecture was known in many cases:

- $g \leq 20$, 22, 23 [Farkas, Lelli-Chiesa, Auel–H., Auel–H.–Larson]
- $g + 1$ or $g + 2 \in \{\mathrm{lcm}(1, \dots, n) \mid n \geq 4\}$ (all expected maximal BN loci have same $\rho \in \{-1, -2\}$) [Eisenbud–Harris, Choi–Kim–Kim]

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Distinguishing Brill–Noether loci

Want to show a non-containment $\mathcal{M}^r_{g,d} \not\subseteq \mathcal{M}^s_{g,e}$.

Let S be a K3 surface with $\mathrm{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}L$, where $H^2 = 2g-2$, $H.L = d$, $L^2 = 2r-2$. Then for smooth irreducible $C \in |H|$, $L|_C$ is a g^r_d.

If $C \in |H|$ has a g^s_e, (A, V), consider the bundle $E_{C,A}$:

$$0 \to E^\vee_{C,A} \to V \otimes \mathcal{O}_S \stackrel{ev}{\to} A \to 0.$$

The Lazarsfeld–Mukai bundle $E_{C,A}$ is closely related to the g^s_e.

- $c_1(E_{C,A}) = H$, $c_2(E_{C,A}) = e$, $\mathrm{rank}(E_{C,A}) = s+1$
- If $\rho(g,s,e) < 0$, there is $\varphi \in \mathrm{End}(E_{C,A})$ that drops rank everywhere, giving

$$0 \to \mathrm{im}\,\varphi \to E_{C,A} \to \mathrm{coker}\,\varphi \to 0$$

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When $\mathcal{M}^r_{g,d}$ is expected maximal,

$$0 \to \operatorname{im}\varphi \to E_{C,A} \to \operatorname{coker}\varphi \to 0$$

is very constrained:

$$c_1(E_{C,A}) = H = c_1(\operatorname{im}\varphi) + c_1(\operatorname{coker}\varphi) = (H-L)+L$$

Lemma (Slogan: g^r_d on H gives $g^{r_1}_{d_1}$ on L and$g^{r_2}_{d_2}$ on $H-L$)

For $\mathcal{M}^r_{g,d}$ expected maximal, and S a K3 as before, $C \in |H|$ carries a g^s_e with $\rho(g,s,e)<0$ if and only if there are integers r_1, r_2, d_1, d_2 such that

- $r_1 + r_2 = s - 1$
- $d_1 + d_2 \le e - d + 2r - 2$
- $0 \le \rho(r, r_1, d_1) < r$
- $0 \le \rho(g-d+r-1, r_2, d_2) < g-d+r-1$

When $\mathcal{M}^s_{g,e}$ is expected maximal, these constraints cannot be satisfied.

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Theorem (Auel–H.–Knutsen)

For $g \geq 3$ and $g \neq 7, 8, 9$, the expected maximal Brill–Noether loci have a component where a general curve admits no further Brill–Noether special divisors.

Questions

- What are the relative positions of Brill–Noether loci in general?
- What is the geometry of curves in given components of Brill–Noether loci?

Thank You!

Questions?