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Brill–Noether Special K3 Surfaces

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Abstract

We answer a question of Mukai and Knutsen on polarized K3 surfaces with a Brill–Noether special curve. Using recent developments in the study of linear systems on curves in polarized K3 surfaces, a bounded version of the Donagi–Morrison conjecture, and a study of Lazarsfeld–Mukai bundles, we prove that a polarized K3 surface of genus $g \leq 19$ is Brill–Noether special if and only if a curve in the polarization class is Brill–Noether special.

Brill–Noether Theory (Curves)

Let C be a curve and $A \in \mathrm{Pic}(C)$ a line bundle. We say A is a g^r_d when $h^0(C, A) = r+1$ and $\deg(A) = d$. The *Clifford index* of A is $\gamma(A) = d - 2r$. The Clifford index of C is $\gamma(C) := \min\{\gamma(A) | A \in \mathrm{Pic}(C), h^0(C,A), h^1(C,A) \geq 2\}$.

The Brill–Noether theorem states that when

$$\rho(g,r,d) = \underbrace{g}_{\text{genus}} - \underbrace{(r+1)}_{h^0(C,A)} \underbrace{(g-d+r)}_{h^0(C,\, \omega_C - A)} \geq 0,$$

then C admits a g^r_d. Therefore $\gamma(C) \leq \lfloor \frac{g-1}{2} \rfloor$.

Moreover, if $\rho(g,r,d) < 0$ then a general curve of genus g has no g^r_d. A line bundle A with $\rho(A) < 0$ is called *Brill–Noether special*, and a curve admitting such a line bundle is also called *Brill–Noether special*.

Brill–Noether Theory (K3 surfaces)

Let (S, H) be a polarized K3 surface of genus g (degree $2g-2$). That is, $H^2 = 2g-2$, and a smooth curve $C \in |H|$ has genus g.

Definition: [Mukai] (S, H) is *Brill–Noether special* if there is a nontrivial $M \neq H \in \mathrm{Pic}(S)$ such that

$$g - h^0(S, M) h^0(S, H - M) < 0.$$

Else (S, H) is called *Brill–Noether general*.

Proposition: If (S, H) is Brill–Noether special, then C is Brill–Noether special.

Proof: Let $M \in \mathrm{Pic}(S)$ make (S, H) Brill–Noether special. Then $M|_C \in \mathrm{Pic}(C)$ is Brill–Noether special.

Theorem [6]: If $\mathrm{Pic}(S) = \mathbb{Z}H$, then $C \in |H|$ is Brill–Noether general.

In particular, $\mathrm{Pic}(S)$ admits a primitive embedding of the lattice [3, 4]

$$\Lambda^r_{g,d} = \begin{array}{c|cc} & H & L \\ \hline H & 2g-2 & d \\ L & d & 2r-2 \end{array}.$$

In the moduli space $\mathcal{K}_g$ of polarized K3 surfaces of genus g, there is a Noether–Lefschetz divisor $\mathcal{K}^r_{g,d}$ parameterizing such polarized K3 surfaces.

Conjecture and Theorem

Brill–Noether special K3 conjecture: Let (S, H) be a polarized K3 surface of genus $g \geq 2$. Then (S, H) is Brill–Noether special if and only if a curve $C \in |H|$ is Brill–Noether special.

Strategy: Suppose that C admits a Brill–Noether special line bundle A. Then find a Donagi–Morrison lift $M \in \mathrm{Pic}(S)$ of A making (S, H) Brill–Noether special.

Theorem [H.]: The conjecture holds in genus $2 \leq g \leq 19$

In genus ≥ 20, we show that a Bounded Donagi–Morrison conjecture implies the Brill–Noether special K3 conjecture.

Lifting Brill–Noether Special Line Bundles

Let $A \in \mathrm{Pic}(C)$ be a Brill–Noether special line bundle. We are interested in finding a lift of A to a line bundle $M \in \mathrm{Pic}(S)$.

Donagi–Morrison Conjecture [1, 8]: Let (S, H) be a polarized K3 surface and $C \in |H|$ be a smooth irreducible curve of genus ≥ 2. Suppose A is a complete basepoint free g^r_d on C such that $d \leq g-1$ and $\rho(g,r,d) < 0$. Then there exists a line bundle $M \in \mathrm{Pic}(S)$ adapted to $|H|$ such that

- $|A|$ is contained in the restriction of $|M|$ to C, and
- $\gamma(M \otimes \mathcal{O}_C) \leq \gamma(A)$.

In particular, $M.H \geq d$. The line bundle M is called a *Donagi–Morrison lift of A*.

Donagi and Morrison verified the Donagi–Morrison conjecture for $r = 1$, and Lelli-Chiesa verified it for $r = 2$ [1, 7] and when $\gamma(A) = \gamma(C)$ [8]. These lifting results prove the Brill–Noether special K3 conjecture when $\gamma(A) \leq \gamma(C)$.

Proposition: Suppose $N \in \mathrm{Pic}(S)$ is a globally generated line bundle and

$$0 \to N \to E_{C,A} \to E \to 0$$

is exact, with E stable. Then $M := \det E$ is a Donagi–Morrison lift of A.

Strong Donagi–Morrison Conjecture [4]: Let (S, H) be a polarized K3 of genus g and A a complete basepoint free g^r_d on $C \in |H|$ with $\rho(g,r,d) < 0$. Then there is a nontrivial globally generated line bundle $N \subset E_{C,A}$ with $E_{C,A}/N$ stable.

By the Proposition above, this implies the Donagi–Morrison conjecture.

Counterexample in genus 19

Let S be a K3 surface of genus 19 with $\mathrm{Pic}(S) = \Lambda^4_{19,16}$. Curves $C_1 \in |H - L|$ and $C_2 \in |L|$ have generic gonality and are Brill–Noether general. Let E_i be the Lazarsfeld–Mukai bundles of the gonality pencils on C_i, and let $E = E_1 \oplus E_2$, which is the Lazarsfeld–Mukai bundle of a g^3_{17} on $C \in |H|$.

It can be shown that E has a filtration of type $0 \subset \mathrm{rk}\, 2 \subset E$, but no injective map from a nontrivial line bundle. Thus the g^3_{17} does not lift to a linear system on S.

However, in this example, C does not have generic Clifford index. In fact, $\gamma(C) \leq 8$.

Modified Donagi–Morrison conjecture

Bounded Strong Donagi–Morrison Conjecture [4]: There is a bound β depending on C and S such that if $d < \beta$, then the Strong Donagi–Morrison conjecture holds.

This has been proven for $r = 2$ by Lelli-Chiesa [7] and for $r = 3$ [2].

Proof Idea

Show that the Donagi–Morrison lift of A makes (S, H) Brill–Noether special! In particular, find restrictions on $M = \det(E)$.

Lazarsfeld–Mukai Bundles

We define a bundle $F_{C,A}$ on S via the short exact sequence

$$0 \longrightarrow F_{C,A} \longrightarrow H^0(C,A) \otimes \mathcal{O}_S \xrightarrow{ev} \iota_*(A) \longrightarrow 0.$$

Dualizing gives $E_{C,A} = F^\vee_{C,A}$ (the LM bundle associated to A on C) sitting in the short exact sequence

$$0 \longrightarrow H^0(C,A)^\vee \otimes \mathcal{O}_S \longrightarrow E_{C,A} \longrightarrow \iota_*(\omega_C \otimes A^\vee) \longrightarrow 0;$$

The LM bundle $E_{C,A}$ is like a lift of A to a vector bundle on S.

Let $E_{C,A}$ be a LM bundle associated to a basepoint free line bundle A of type g^r_d on $C \subset S$, then:

- $\mathrm{rk} = r+1$, $c_1 = H = [C]$, $c_2 = d$
- $E_{C,A}$ is globally generated off the base locus of $\iota_*(\omega_C \otimes A^\vee)$
- If $\rho(A) < 0$, then $E_{C,A}$ is not stable

Restrictions on Lifts from Stable Quotients

Let A be a Brill–Noether special line bundle on $C \in |H|$, and assume the Strong Donagi–Morrison conjecture. Let $E = E_{C,A}/N$ be stable. We would like to show that $M = \det(E)$ makes (S, H) Brill–Noether special.

Let $M^2 = 2r' - 2$ and $M.H = d'$.

Lemma: We may assume $\gamma(A) > \gamma(C) = \left\lfloor \frac{g-1}{2} \right\rfloor$.

Proof: The lifting results of Knutsen [5] and Lelli-Chiesa [8] suffice to show that (S, H) is Brill–Noether special when $\gamma(A) = \gamma(C) \leq \left\lfloor \frac{g-1}{2} \right\rfloor$.

We study restrictions on $c_2(E)$ and M coming from stability of E and show that

- $r' \geq r$
- "$\gamma(M)$" $:= d' - 2r' \leq \gamma(A)$
- $\rho(g, r', d') \leq \rho(g, r, d) < 0$

Theorem: $\rho(g, r', d') < 0$.

Theorem: The Strong Donagi–Morrison conjecture implies the Brill–Noether special K3 conjecture.

Proof: As $h^2(S, M) = h^2(S, N) = 0$, $h^0(S, M) \geq r'+1$ and $h^0(S, H-M) \geq g - d' + r'$. Hence M makes (S, H) Brill–Noether special.

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