Brill–Noether theory via K3 surfaces

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SPOILER ALERT!

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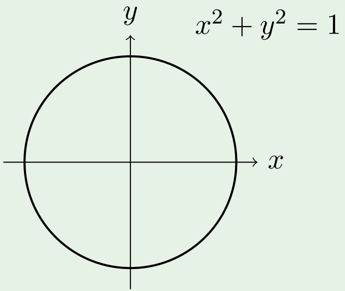
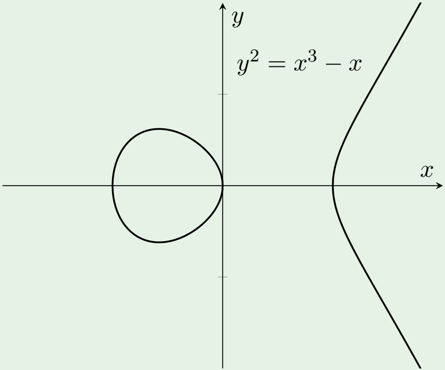
Theorem (Crucial Result)

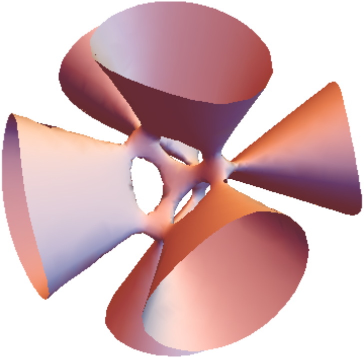
There are no solutions to $4y^2 \equiv 2 \pmod{13}$.

In algebraic geometry, we study *varieties*, which are spaces defined as the solutions to polynomial equations.





But what if you don't have equations? Can you still talk about varieties?



Algebraic Geometry

But what if you don't have equations? Can you still talk about varieties?

- Are there any constraints on what those polynomial equations can be?
- What controls those constraints?
- How can we find them?

Curves

Brill–Noether theory studies the ways that curves can be defined by polynomials.

Definition

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Definition

A *divisor* on C is a formal $\mathbb{Z}$-linear combination of points of C.

If $D = a_1p_1 + a_2p_2 + \cdots + a_np_n$, we say D has degree $d = \sum_{i=1}^{n} a_i$.

A divisor is associated to a *line bundle* $\mathcal{O}(D)$ which tells us the functions with poles or zeros along D.

Riemann–Roch

Riemann–Roch Problem

How many functions on C have certain poles and zeros?

$$H^0(C,D) = \{f : C \to \mathbb{C} \mid f \text{ has poles and zeros dictated by } D\}$$

$$h^0(C,D) = \dim H^0(C,D)$$

How does $\deg(D)$ influence $h^0(C,D)$?

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Theorem (Riemann–Roch Theorem)

$$h^0(C, D) - h^1(C, D) = d - g(C) + 1$$

The genus $g(C)$ tells us a lot about the geometry of C.

The Nicest Curve

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Take the point $[0 : 1] \in \mathbb{P}^1$, and make a divisor $D = 3[0 : 1]$.

$H^0(\mathbb{P}^1, D)$ is generated (over $\mathbb{C}[x, y]$) by the functions $\left\{\frac{x^3}{x^3}, \frac{x^2y}{x^3}, \frac{xy^2}{x^3}, \frac{y^3}{x^3}\right\}$.

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We'll think of these as the functions x^3, x^2y, xy^2, y^3. (Which is what we get after we multiply by x^3)

Maps to $\mathbb{P}^r$

On $\mathbb{P}^1$, we can consider the cubic forms x^3, x^2y, xy^2, y^3, which give us a map

$$\mathbb{P}^1 \to \mathbb{P}^3,\ [x:y] \mapsto [x^3 : x^2y : xy^2 : y^3].$$

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We can use the functions in $H^0(C, D)$ to cook up maps to $\mathbb{P}^r$.

Say $f_0, f_1, \ldots, f_r \in H^0(C, D)$ have no common zeros.
We define a map $C \to \mathbb{P}^r$ that is given by

$$\varphi_{\{f_i\}} : C \to \mathbb{P}^r,\ p \to [f_0(p) : f_1(p) : \cdots : f_r(p)]$$

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Definition

A *(complete) linear series* on C is a basis of $H^0(C, D)$.
We say it is of type g^r_d if $h^0(C, D) = r + 1$ and $\deg(D) = d$.

What embeddings do I have?



Brill–Noether Theory

Brill–Noether theory studies the ways curves can map to projective space. So it studies linear series on curves.

Recall that a linear system is a g^r_d if it gives a map $C \to \mathbb{P}^r$ of degree d.

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Questions

If C has genus g,

- what g^r_d's does it have?
- what is the minimal k such that C has a g^1_k?
  - The minimal k is called the *gonality* of C, it measures how far C is from being $\mathbb{P}^1$.
- and has a g^r_d, what other g^s_e does it have/not have?

Smooth Plane curves

The kinds of linear systems a curve has is constrained by its geometry.

Theorem (Genus–Degree Formula)

Let C be a smooth plane curve of degree d (the zero set of a polynomial $f(x, y)$ of degree d). Then

$$g(C) = \frac{(d-1)(d-2)}{2}.$$

Example

In particular, a smooth plane cubic (degree 3) has genus $\frac{(2)(1)}{2} = 1$.

Clifford index

Theorem (Clifford's Theorem)

Let D be a g^r_d with $r \geq 0$ and $g - d + r \geq 1$, then

$$\gamma(D) := d - 2r \geq 0.$$

Equality holds if and only if $D = 0$ or C has a g^1_2 and D is a multiple of it.

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Definition

The *Clifford index* of a curve C is the integer

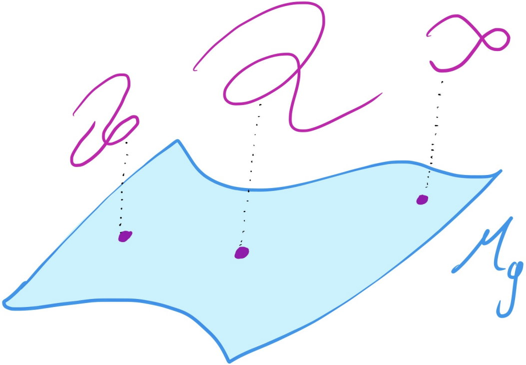
$$\min\left\{\gamma(D) \mid h^0(C,D), h^1(C,D) \geq 2\right\}.$$

Theorem (Clifford's Theorem)

$\gamma(C) \geq 0$ with equality if and only if C has a g^1_2.

Moduli space of curves

Curves of genus g can be packaged together into a parameter space $\mathcal{M}_g$ of dimension $3g-3$.



What do the curves in $\mathcal{M}_g$ with a g^r_d look like?

Brill–Noether loci

Definition

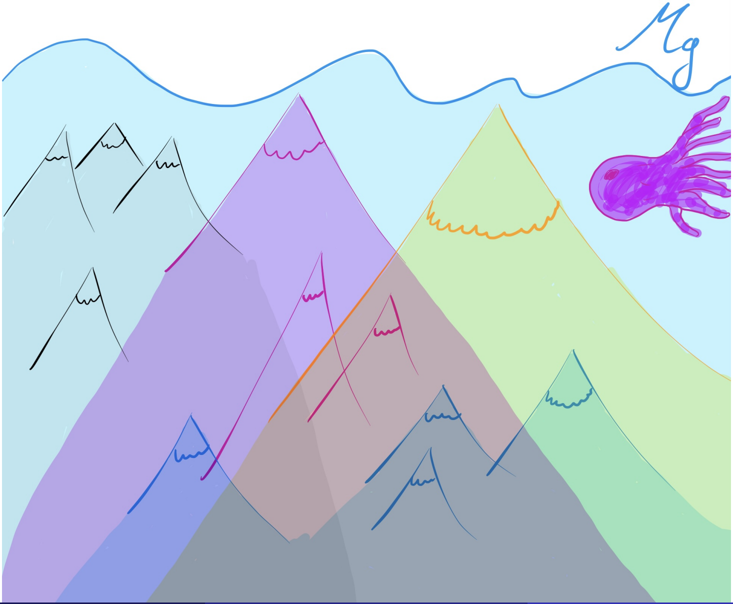
$$\mathcal{M}^r_{g,d} := \{C \in \mathcal{M}_g \mid C \text{ has a } g^r_d\}$$

is called a Brill–Noether locus.

Questions

- Is $\mathcal{M}^r_{g,d}$ non-empty?
- What's the geometry of $\mathcal{M}^r_{g,d}$?
- How do different Brill–Noether loci overlap?

Deep Sea Diving



Brill–Noether theorem

Definition

The *Brill–Noether number* is

$$\rho(g,r,d) = \underbrace{g}_{\text{genus}(C)} - \underbrace{(r+1)}_{h^0(C,D)}\underbrace{(g-d+r)}_{h^1(C,D)}.$$

Theorem (Brill–Noether theorem)

If C is a general curve in $\mathcal{M}_g$ and $\rho(g,r,d) \geq 0$, then C has a g^r_d.
If $\rho(g,r,d) < 0$, then C has no g^r_d.

So for $\rho(g,r,d) < 0$, $\mathcal{M}^r_{g,d} \subsetneq \mathcal{M}_g$, and such curves are called *Brill–Noether special*. We focus on these.

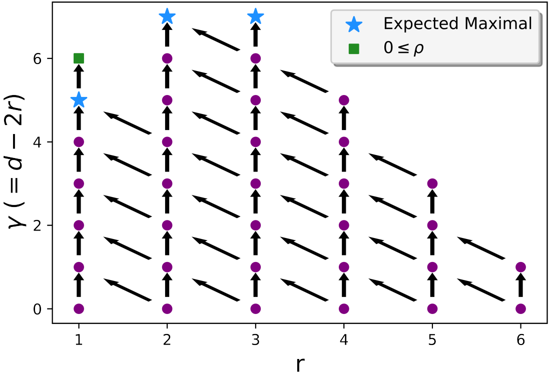
In fact, the expected codimension of $\mathcal{M}^r_{g,d}$ is $-\rho$.

Trivial Containments

Question

How do $\mathcal{M}^r_{g,d}$ and $\mathcal{M}^s_{g,e}$ overlap?

- $\mathcal{M}^r_{g,d} \subseteq \mathcal{M}^r_{g,d+1}$
- $\mathcal{M}^r_{g,d} \subseteq \mathcal{M}^{r-1}_{g,d-1}$



Maximal Brill–Noether loci

Definition

We say that $\mathcal{M}^r_{g,d}$ is *expected maximal* if $\rho(g,r,d) < 0$ and it is not trivially contained in another Brill–Noether locus.

Maximal Brill–Noether loci conjecture

For $g \geq 3$, the expected maximal Brill–Noether loci are maximal (not contained in each other), except for genus $7, 8, 9$.

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Theorem (Auel–H.)

The Maximal Brill–Noether loci conjecture holds in genus 3–19, 22, 23.

For example, in genus 14, the expected maximal loci are $\mathcal{M}^1_{14,7}$, $\mathcal{M}^2_{14,11}$, and $\mathcal{M}^3_{14,13}$

Let's prove it!

We want to show each of the loci $\mathcal{M}^1_{14,7}$, $\mathcal{M}^2_{14,11}$, $\mathcal{M}^3_{14,13}$ are not contained in one another.

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- $\rho(14, 1,\ 7) = -2$, so $\dim \mathcal{M}^1_{14,7} = 37$
- $\rho(14, 2, 11) = -1$, so $\dim \mathcal{M}^2_{14,11} = 38$
- $\rho(14, 3, 13) = -2$, so $\dim \mathcal{M}^3_{14,13} = 37$

So we have $\mathcal{M}^2_{14,11} \not\subseteq \mathcal{M}^1_{14,7}$ and $\mathcal{M}^2_{14,11} \not\subseteq \mathcal{M}^3_{14,13}$.

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We can find $C \in \mathcal{M}^3_{14,13}$ with gonality 8, hence $\mathcal{M}^3_{14,13} \not\subseteq \mathcal{M}^1_{14,7}$.

Let's prove it!

In recent years, there has been a surge of results concerning the Brill–Noether theory for curves of *fixed gonality*.

Theorem (Coppens–Martens, Pflueger, Jensen–Ranganathan, Larson, Vogt,. . .)

Let C be a general curve of gonality k, and $r' = \min\{r, g-d+r-1\}$, then

$$\dim\{g^r_d\text{'s on } C\} = \rho_k(g,r,d) := \max_{\ell\in\{0,\dots,r'\}} \rho(g, r-\ell, d) - \ell k.$$

By considering curves $C \in \mathcal{M}^1_{14,7}$ with gonality 7, we can show $\mathcal{M}^1_{14,7} \not\subseteq \mathcal{M}^2_{14,11}$ and $\mathcal{M}^1_{14,7} \not\subseteq \mathcal{M}^3_{14,13}$.

Last one!

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Last one!

It remains to show that $\mathcal{M}^3_{14,13} \not\subseteq \mathcal{M}^2_{14,11}$. We just need to find *one* curve!

We'll find a genus 14 curve with a g^3_{13} but no g^2_{11}.

K3 surfaces

A K3 surface is a (sm. proj.) variety S of dimension 2 with $K_S = 0$ and $H^1(S, \mathcal{O}) = 0$.
For us, the important fact will be that $\mathrm{Pic}(S)$ is a *lattice*.

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For us, the important fact will be that $\mathrm{Pic}(S)$ is a *lattice*.

$$\mathrm{Pic}(S) = \begin{array}{c|cc} & H & L \\ \hline H & 26 & 13 \\ L & 13 & 4 \end{array}$$

For $C \in |H|$ $\mathrm{genus}(C) = \frac{H^2+2}{2}$, $\deg L|_C = L.H$, and $h^0(C, L|_C) - 1 = \frac{L^2+2}{2}$.

So C has genus 14 and $L|_C$ is a g^3_{13}!

What happens if C has a g^2_{11}?

Donagi–Morrison conjecture

If $C \subset S$ has a Brill–Noether special line bundle, is it the restriction of a line bundle on S?

Conjecture (Donagi–Morrison, Lelli-Chiesa)

Let (S, H) be a polarized K3 surface and $C \in |H|$ a smooth irreducible curve of genus $g \geq 2$. Suppose A is a basepoint free g^r_d on C such that $d \leq g-1$ and $\rho(g,r,d) < 0$. Then there exists a line bundle $M \in \mathrm{Pic}(S)$ adapted to $|H|$ such that $|A|$ is contained in the restriction of $|M|$ to C and $\gamma(M|_C) \leq \gamma(A)$.

We call M a *Donagi–Morrison* lift of A.

This turns out to be false in general. In fact there is a counterexample to lifting g^3_d's in genus 19!

Donagi–Morrison conjecture

So what could be true?

Bounded Donagi–Morrison conjecture

There is a bound β depending on S and C, such that if $d \leq \beta$, then the Donagi–Morrison conjecture holds.

Donagi–Morrison conjecture

What is known?

Theorem

The (bounded) Donagi–Morrison conjecture holds when:

- $r = 1$ *(Saint-Donat, Reid, Donagi–Morrison)*
- $r = 2$ *(Lelli-Chiesa)*
- $\gamma(A) = \gamma(C)$ *(Green–Lazarsfeld, Lelli-Chiesa)*

Theorem (H.)

The bounded Donagi–Morrison conjecture holds when $r = 3$*, and the bounds are explicit.*

Back to $\mathcal{M}^3_{14,13} \not\subseteq \mathcal{M}^2_{14,11}$

Let (S, H) be a polarized K3 surface with

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Then $L|_C$ is a g^3_{13}. So $C \in \mathcal{M}^3_{14,13}$.

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If $M = xH + yL \in \mathrm{Pic}(S)$, then $26x^2 + 26xy + 4y^2 = 2$.

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Theorem (Crucial Result)

There are no solutions to $4y^2 \equiv 2 \pmod{13}$.

How do we find lifts?

The ideas go back to Lazarsfeld's proof of the Brill–Noether theorem using K3 surfaces.
Let A be a g^r_d on C.

Construction of Lazarsfeld–Mukai Bundles

$$H^0(C,A)\otimes\mathcal{O}_S \longrightarrow A \longrightarrow 0$$

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Construction of Lazarsfeld–Mukai Bundles

$$\begin{tikzcd}
0 \arrow[r] & F_{C,A} \arrow[r] & H^0(C,A) \otimes \mathcal{O}_S \arrow[r] \arrow[d, squiggly, "\text{dualize and remember } \mathcal{E}xt"] & A \arrow[r] & 0 \\
0 \arrow[r] & H^0(C,A)^\vee \otimes \mathcal{O}_S \arrow[r] & E_{C,A} \arrow[r] & \omega_C \otimes A^\vee \arrow[r] & 0
\end{tikzcd}$$

Lazasfeld–Mukai Bundles

The bundle $E_{C,A}$ is a vector bundle on S called the *Lazarsfeld–Mukai* bundle associated to (C, A).

Properties of $E_{C,A}$

- $\operatorname{rk} E_{C,A} = h^0(C, A) = r + 1$
- $c_1(E_{C,A}) = [C] = H$
- $c_2(E_{C,A}) = \deg A = d$
- $2 - 2\rho(g, r, d) = 2h^0(S, \mathcal{E}nd(E_{C,A})) - h^1(S, \mathcal{E}nd(E_{C,A}))$

Proposition

If there is a globally generated line bundle $N \subset E_{C,A}$ such that $E_{C,A}/N$ is torsion-free, then $M = \det(E_{C,A}/N)$ is a Donagi–Morrison lift of A.

The trouble is finding N.

Stability of sheaves on K3 surfaces

Let (S, H) be a polarized K3 surface, and E a vector bundle on S.

Definition

The slope of E is

$$\mu(E) := \frac{c_1(E).H}{\operatorname{rk} E}$$

Definition

E is called *(semi)stable* if for every proper subsheaf $N \subset E$ of smaller rank we have

$$\mu(N)(\leq)\mu(E).$$

Otherwise, we say E is unstable.

Fact

If $\rho(g, r, d) < 0$, then $E_{C,A}$ is not stable.

Filtrations

Suppose we knew the following fact:

Dream Theorem

If $E_{C,A}$ is not stable, then it has a sub-line bundle so that $E_{C,A}/N$ is torsion-free.

That may not always be true. But:

Roughly True

If $E_{C,A}$ is not stable, it has a filtration

$$0 \subseteq E_1 \subseteq E_2 \subseteq \cdots \subseteq E_{r+1} = E_{C,A}$$

such that E_{i+1}/E_i is torsion free.

So what kind of filtrations does $E_{C,A}$ have?

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So what kind of filtrations does $E_{C,A}$ have fi A is a g^3_d?

Since $r = 3$, $\mathrm{rk}\, E_{C,A} = 4$.

- $1 \subset 4$
- $2 \subset 4, \quad 3 \subset 4,$
  $1 \subset 2 \subset 4, \quad 1 \subset 3 \subset 4, \quad 2 \subset 3 \subset 4,$
  $1 \subset 2 \subset 3 \subset 4$

We want to eliminate all options except $1 \subset 4$.

Lift off!

Let $E_{C,A}$ be the Lazarsfeld–Mukai bundle associated to a Brill–Noether special line bundle $A \in \mathrm{Pic}(C)$ of type g^3_d.

Theorem (H.)

Let (S, H) be a polarized K3 surface of genus $g \neq 2, 3, 4, 8$, and $C \in |H|$ a smooth irreducible curve of Clifford index γ. Let

$$m := \left\{ D^2 \;\mid\; D \in \mathrm{Pic}(S),\; D^2 \geq 0,\; D \textit{ is effective} \right\},$$

$$\mu := \min \left\{ \mu(D) \;\mid\; D \in \mathrm{Pic}(S),\; D^2 \geq 0,\; \mu(D) > 0 \right\}.$$

If $d < \min \left\{ \frac{5\gamma}{4} + \frac{\mu + m + 9}{2}, \frac{5\gamma}{4} + \frac{m + 10}{2}, \frac{3\gamma}{2} + 5, \frac{\gamma + g - 1}{2} + 4 \right\}$,

then $E_{C,A}$ only has a $1 \subset 4$ filtration.

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- All of you!

Thank You!

Questions?