Refined Brill–Noether Theory

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Classical Brill–Noether theory

Brill–Noether theory studies linear series on (smooth) algebraic curves. Let C be a smooth curve.

By a g^r_d, we mean a linear series of dimension r and degree d.

(Base point free g^r_d gives a (non-degenerate) map $C \to \mathbb{P}^r$ of degree d.)

Question (Brill–Noether Theory)

What g^r_d's does C have?

Brill–Noether loci

Brill–Noether Theorem [Eisenbud, Fulton, Gieseker, Griffiths, Harris, Kempf, Kleiman, Lazarsfeld]

A general curve $C \in \mathcal{M}_g$ admits a g^r_d if and only if

$$\rho(g, r, d) := g - (r+1)(g - d + r) \geq 0.$$

Thus when $\rho(g, r, d) < 0$, the *Brill–Noether locus*

$$\mathcal{M}^r_{g,d} := \{C \in \mathcal{M}_g \text{ admitting a } g^r_d\} \text{ is a subvariety of } \mathcal{M}_g.$$

Question (Refined Brill–Noether theory)

For a "general" curve in $\mathcal{M}^r_{g,d}$, what g^s_e's does it have?

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- When $r = 1$, $\mathcal{M}^1_{g,d}$ is irreducible.
  - Refined Brill–Noether theory for curves of fixed gonality (answers question for $r = 1$) [Pflueger, Jensen–Ranganathan, H. Larson, Larson–Larson–Vogt]
- For $r \geq 2$, $\mathcal{M}^r_{g,d}$ can have multiple components of various dimensions!
- Curves in different components can behave very differently! ($\mathcal{M}^3_{10,9}$)

Relative positions of Brill–Noether loci give a coarse answer.

We have trivial containments:

- $\mathcal{M}^r_{g,d} \subset \mathcal{M}^r_{g,d+1}$ by adding a basepoint
- $\mathcal{M}^r_{g,d} \subset \mathcal{M}^{r-1}_{g,d-1}$ by subtracting a non-basepoint

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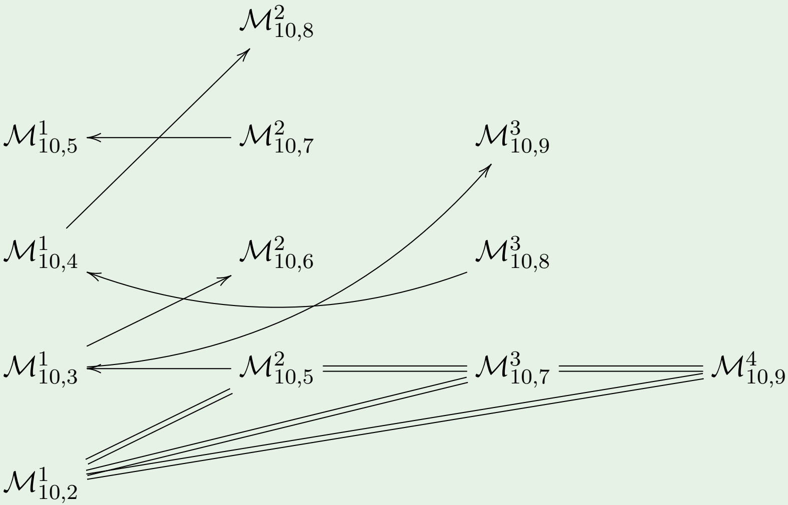
Brill–Noether loci in genus 10 (omitting trivial containments ↑ & ↖)



Relative positions of Brill–Noether loci

Theorem (H. 2025)

The relative positions of Brill–Noether loci in genus $g \leq 6$ are given by trivial containments.

For $7 \leq g \leq 12$, the relative positions are identified.

Genus 13

The relative position of $\mathcal{M}^4_{13,12}$???

Predicting relative positions of Brill–Noether loci with K3s

Let $g \geq 3$, $r \geq 1$, and $2 \leq d \leq g-1$, $4(g-1)(r-1) - d^2 < 0$.

Let (S, H) be a polarized K3 surface with $\mathrm{Pic}(S) = \Lambda^r_{g,d}$,
where $\Lambda^r_{g,d}$ is the lattice $\mathbb{Z}[H] \oplus \mathbb{Z}[L]$ with intersection matrix

$$\begin{bmatrix} H^2 & H.L \\ H.L & L^2 \end{bmatrix} = \begin{bmatrix} 2g-2 & d \\ d & 2r-2 \end{bmatrix}.$$

For $C \in |H|$ smooth irred., $C \in \mathcal{M}^r_{g,d}$ ($|\mathcal{O}_C(L)|$ is a base point free g^r_d).

Philosophy

Such K3s detect the behavior of the "most general" curves $\in \mathcal{M}^r_{g,d}$.

Destabilizing filtrations

C admits a g^s_e exactly when S admits the Lazarsfeld–Mukai bundle E_{C,g^s_e}.

$$0 \to E_{C,A}{}^\vee \to V \otimes \mathcal{O}_S \xrightarrow{ev} A \to 0, \quad \text{where } g^s_e = (A, V)$$

When $\rho(g,s,e) < 0$, E_{C,g^s_e} is unstable, so has some destabilizing filtration

$$0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E_{C,g^s_e}$$

may assume E_{i+1}/E_i stable. There are many constraints, e.g. the slopes $\mu_{i,j} = \mu_H(E_i/E_{i-j})$ also fit into a Gelfand–Tsetlin pattern ($\mu_{i,j} \geq \mu_{i+1,j+1} \geq \mu_{i+1,j}$).

$c_2(E_n)$ obtained recursively from c_1 and c_2 of the factors.

Since $\mathrm{Pic}(S) = \Lambda^r_{g,d}$ is fixed, an assignment of c_1's to the E_i gives a lower bound on $e = c_2(E_{C,g^s_e})$ (if all bounds $> e$ then E_{C,g^s_e} does not exist!)

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Example $\mathcal{M}^2_{100,d} \overset{?}{\subset} \mathcal{M}^3_{100,e}$

Does a K3 surface S with $\mathrm{Pic}(S) = \Lambda^2_{100,d}$ admit a bundle $E = E_{C,g^3_e}$?
For $d \geq 52$, checking assignments of c_1's, no such bundle exists!

$d = 51$

There is a destabilizing filtration $E_1 \subset E$ with $\mathrm{rk}\, E_1 = 2$, $c_1(E_1) = H - L$ and $c_2(E_1) = 26$.
In fact, $E_{H-L,g^1_{26}} \oplus E_{L,g^1_2}$ is a Lazarsfeld–Mukai bundle of type g^3_{77} ($\mathcal{M}^3_{100,77}$ is maximal).

So we predict a containment $\mathcal{M}^2_{100,51} \overset{?}{\subset} \mathcal{M}^3_{100,77}$.

$d < 50$

Many destabilizing filtrations appear as d decreases
$38 \leq d \leq 50$: ranks $2 \subset E$, $c_1(E_1) = H - L$;
$d = 37$: ranks $1 \subset E$, $c_1(E_1) = H - 2L$;
$21 \leq d \leq 36$: ranks $1 \subset 2 \subset E$, $c_1(E_1) = H - 2L$, $c_1(E_2) = H - L$;
$d = 20$: Over 2000 filtrations! $d < 20$: No such K3s.

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K3-expected non-containments

For large g and d, the smallest bounds $c_2(E_n)$ appear to come from destabilizing filtrations of the form $E_1 \subset E_n$ with E_n/E_1 stable and $\mathrm{rk}(E_1) = 2, c_1(E_1) = H - L$.

Conjecture

For g and d sufficiently large, $d, e \leq g - 1$, and $2 \leq r < s$,

$$\text{if } e < d - 2r + s + \frac{g - d + r + 1}{2} + \frac{(s-2)(r-1) - 1}{s - 1}, \text{ then } \mathcal{M}^r_{g,d} \not\subseteq \mathcal{M}^s_{g,e}.$$

Can be checked numerically, given particular values of g, r, d, s, e, but showing that no other filtrations exist is difficult (also difficult to show that this filtrations gives tightest bound).

K3-expected containments

Conversely, when d is slightly smaller, a destabilizing filtration may exist for E_{C,g^s_e}.

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Definition

The potential containment $\mathcal{M}^r_{g,d} \overset{?}{\subset} \mathcal{M}^s_{g,e}$ is called *K3-expected* if such a K3 surface S with $\mathrm{Pic}(S) = \Lambda^r_{g,d}$ admits a vector bundle E_{C,g^s_e}.

Philosophy

(In some range) K3-expected containments hold.

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Other sources of (non)-containments

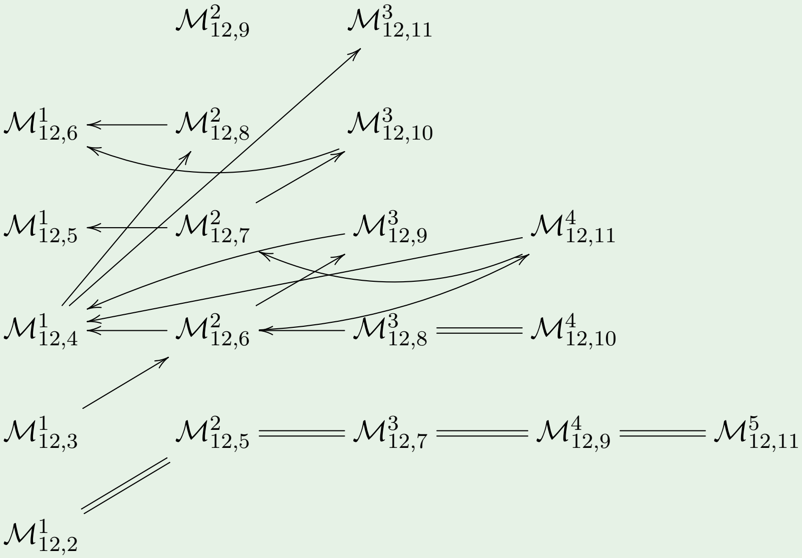
Containments

- Curves on Hirzebruch surfaces [Larson–Vemulapalli]
- Highly secant hyperplanes to curves in $\mathbb{P}^r$
- Castelnuovo curves
- Low Clifford index and Castelnuovo–Severi inequality

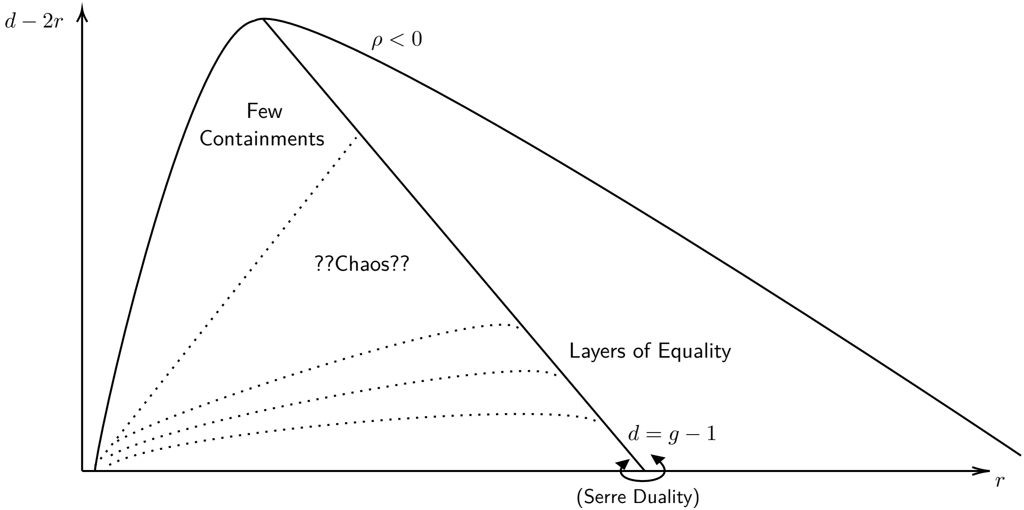
Non-containments

- Covers of curves (bi-elliptic curves play a large role in distinguishing loci with $d - 2r = 1$ and $d - 2r = 2$)
- Chains of elliptic curves and admissible fillings of tableaux [Pflueger, Teixidor i Bigas]
- Castelnuovo curves
- Gonality of nodal plane curves

Brill–Noether loci in genus 12 (omitting trivial containments ↑ & ↖)



Geology of Brill–Noether loci



Thank You!

Questions?