BRILL–NOETHER THEORY VIA K3 SURFACES

A Thesis
Submitted to the Faculty
in partial fulfillment of the requirements for the
degree of

Doctor of Philosophy

in

Mathematics

by Richard Haburcak

Guarini School of Graduate and Advanced Studies
Dartmouth College
Hanover, New Hampshire

April 2023

Examining Committee:

Asher Auel, Chair

John Voight

David Webb

Andreas Leopold Knutsen

F. Jon Kull, Ph.D.
Dean of the Guarini School of Graduate and Advanced Studies

Abstract

Brill–Noether theory studies the different projective embeddings that an algebraic curve admits. For a curve with a given projective embedding, we study the question of what other projective embeddings the curve can admit. Our techniques use curves on K3 surfaces. Lazarsfeld's proof of the Gieseker–Petri theorem solidified the role of K3 surfaces in the Brill–Noether theory of curves. In this thesis, we further the study of the Brill–Noether theory of curves on K3 surfaces.

We prove results concerning lifting line bundles from curves to K3 surfaces. Via an analysis of the stability of Lazarsfeld–Mukai bundles, we deduce a bounded version of a conjecture of Donagi–Morrison concerning when a Brill–Noether special line bundle of rank 3 on a curve on a polarized K3 surface lifts to a line bundle on the K3 surface. In joint work with Asher Auel, we also present a strategy for distinguishing Brill–Noether loci by studying the lifting of linear systems on curves in polarized K3 surfaces, which motivates a conjecture identifying the maximal Brill–Noether loci. Using our new lifting results, we prove cases of the maximal Brill–Noether loci conjecture. We also investigate the Brill–Noether theory of K3 surfaces and verify cases of a conjecture of Knutsen and Mukai concerning the Picard groups of K3 surfaces with Brill–Noether special curves.

Acknowledgments

I would like to thank Asher Auel for being my advisor and for his enthusiasm and support for research, learning, teaching, and my non-math endeavors. Over the last few years, we've edited a paper while getting lost on a road trip, convinced ourselves of absolutely wrong math, and enjoyed many cups of tea together. I could not imagine a kinder and more respectful mentor. I feel that we become collaborators and friends working on problems together, and every meeting was a pleasure.

I would also like to thank my committee, Asher Auel, Andreas Leopold Knutsen, John Voight, and David Webb for reading my thesis, and for their insight into research.

I want to thank John Voight for his support, both mathematical and personal, for consistently reminding me that examples are really important, and for his patience reading Hartshorne early in the mornings.

I am indebted to many people for their help in research. I would like to thank Gabi Farkas for elucidating connections between Brill–Noether loci and the birational geometry of $\mathcal{M}_g$ and for interesting discussions on future directions for research; David Jensen and Nathan Pflueger for very helpful conversations about Brill–Noether theory; Hannah Larson for pointing out the exceptional case of genus 9 and for many other interesting discussions; Margherita Lelli-Chiesa for explaining technical aspects of her work and helping me when I was stuck on the question of Brill–Noether special K3 surfaces; Andreas Leopold Knutsen for helpful comments on the Donagi–Morrison conjecture; Isabel Vogt for her support and enthusiasm for these projects, comments on a draft of the paper, and pointing out recent relevant literature; and John Voight for helpful computations and comments on a draft of the paper.

I would like to thank many of my teachers for all of the work they did that helped me get here. Ben Delorio, who told me to slow down and use pencils. Bong Lian, who said it was okay to take advanced math classes and who then added weeks of background material to classes so that I could keep up. David Webb, for his categorical perspectives on algebra that I hope to fully appreciate one day.

I want to thank Tracy Moloney, for her support, for keeping the department running, and for somehow always having time for a chat.

Finally, and most importantly, I would like to thank my friends and family. Lizzie Buchanan, Juanita Duque Rosero, Steve Fan, Grant Molnar, and Alex Wilson for all of the support they gave me throughout our years as students together, for struggling through classes and exams together, and for being the "usual suspects" when the department needed some fixing. Lizzie Buchanan and Alex Wilson for their friendship, kind words, and endless support. Juanita Duque Rosero and Alex Wilson for their guidance with the Directed Reading Program, as well as Ben Adenbaum and Brian Mintz for keeping the vision of the Directed Reading Program going. All of the people at DANTS throughout the years, making attending seminars fun. Sarah Frei and Jack Petok for so many ski trips and fun conversations both mathematical and not. And my parents, Csilla and Marian, who didn't sufficiently convince me that being an academic was hard, for all of their love.

I would also like to acknowledge that this thesis is based upon work partially supported by the National Science Foundation under Grant No. DMS-2200845 and supported by a grant from the Simons Foundation (712097, to Asher Auel).

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Chapter 1

Introduction

Algebraic geometry is the study of the set of solutions to a system of polynomial equations, called a variety. Varieties of dimension 1, algebraic curves, already exhibit a range of interesting behavior. For example, genus 0 curves are the projective line $\mathbb{P}^1$ and smooth conics, and genus 1 curves are elliptic curves, which have been central objects of study in number theory and algebraic geometry. The geometry of higher genus curves becomes more intricate, and moduli spaces of curves of higher genus have fascinated mathematicians since the 19^{th} century. In particular, writing down an explicit curve of high genus is quite difficult, unless the curve belongs to very special families. The moduli space $\mathcal{M}_g$ of genus g curves is an object of great interest in topology and geometry, and special families of curves inside the moduli space can shed light on the geometry of $\mathcal{M}_g$.

Classical Brill–Noether theory concerns the study of linear systems on curves, which roughly speaking, correspond to embeddings of the curve in projective space. Given a linear system on a curve C, we say it is of type g^r_d if the linear system induces a morphism $C \to \mathbb{P}^r$ of degree d. The presence of these linear systems provide information about geometric and arithmetic properties of the curve. Brill–Noether theory is in some sense studying how the geometry of curves constrains the way they can be represented by equations. Classically, questions of interest are describing the space $G^r_d(C)$ of all g^r_ds on a curve C, how the genus of the curve affects these spaces, and how the presence of linear systems affects the geometry of curves. The Brill–Noether theorem, recalled in Theorem 2.49, states that when the *Brill–Noether number* $\rho(g,r,d) = g-(r+1)(g-d+r)$ is non-negative, the general curve of genus g has a g^r_d, that is, $G^r_d(C)$ is nonempty. Moreover, much is known about the geometry of $G^r_d(C)$ [4]. However, when $\rho(g,r,d) < 0$, $G^r_d(C)$ is empty for a general curve. This means that in $\mathcal{M}_g$, the curves which do admit a g^r_d with $\rho(g,r,d) < 0$ form a proper subvariety called a *Brill–Noether locus*, denoted $\mathcal{M}^r_{g,d}$. Such curves and linear systems are called *Brill–Noether special*.

Lazarsfeld's proof of the Brill–Noether–Petri theorem using smooth curves on K3 surfaces solidified the role of K3 surfaces in Brill–Noether theory. The proof also introduced Lazarsfeld–Mukai bundles, which are vector bundles on the K3 surface associated to linear systems on curves, and highlighted the utility of restricting to curves on K3 surfaces to answer questions about curves in general.

The study of special divisors on curves was considered by Saint-Donat, Reid, and others [21, 32, 44, 46, 56, 57, 61, 75, 77]. Following classical work on linear systems of type g^1_d, Harris and Mumford conjectured that the gonality of curves on K3 surfaces should remain constant in a linear system. More generally, the propagation of special divisors in linear systems on K3 surfaces was of interest. A counterexample by Donagi and Morrison showed that this could not hold in general. The conjecture was modified by Green, who conjectured that the Clifford index of curves should remain constant in a linear system on a K3 surface. This was proven by Green and Lazarsfeld [32] where they expanded the use of Lazarsfeld–Mukai bundles to study curves on K3 surfaces. The study of curves on K3 surfaces via Lazarsfeld–Mukai bundles has a rich history, including work by Farkas, Hoff, Lelli-Chiesa, Knutsen, and others [3, 39, 55, 56, 57].

These results were later improved by Lelli-Chiesa [57] where they also prove that the Clifford index is computed by the restriction to C of a line bundle on the K3 surface.

Classically, there are results of Saint-Donat [77] and Reid [75] concerning when a linear system of type g^1_d on a curve C on a polarized K3 surface is the restriction of a line bundle to C. These were extended by Donagi and Morrison [21], and they conjectured that any Brill–Noether special linear system on such curves is the restriction of a linear system on the K3 surface, see Section 4.2 for relevant definitions.

Conjecture 1.1 (Donagi–Morrison Conjecture, [57] Conjecture 1.3)**.** *Let (S, H) be a polarized K3 surface and $C \in |H|$ be a smooth irreducible curve of genus ≥ 2. Suppose A is a complete basepoint free g^r_d on C such that $d \leq g-1$ and $\rho(g, r, d) < 0$. Then there exists a line bundle $M \in \operatorname{Pic}(S)$ adapted to $|H|$ such that $|A|$ is contained in the restriction of $|M|$ to C and $\gamma(M|_C) \leq \gamma(A)$.*

In [21], Donagi and Morrison proved Conjecture 1.1 for linear systems of rank 1. Via an analysis of the stability of Lazarsfeld–Mukai bundles of rank 3, Lelli-Chiesa gave a proof for linear systems of rank 2 [56]. Building on these ideas, we give a result for linear systems of rank 3 [6], see Section 5.2 for more details.

Theorem 1.2 ([6])**.** *Let (S, H) be a polarized K3 surface of genus $g \neq 2, 3, 4, 8$, $C \in |H|$ a smooth irreducible curve of Clifford index γ, and A a g^3_d on C. Let*

$$m := \min\{D^2 \mid D \in \operatorname{Pic}(S),\ D^2 \geq 0,\ \textit{and } D \textit{ is effective}\}$$

(in particular, there are no curves of genus less than $\frac{m+2}{2}$ on S), and let

$$\mu := \min\{D.H \mid D \in \operatorname{Pic}(S),\ D^2 \geq 0,\ \textit{and } D.H > 0\}.$$

If

$$d < \min\left\{\frac{5}{4}\gamma + \frac{\mu}{2} + \frac{m}{2} + \frac{9}{2},\ \frac{5}{4}\gamma + \frac{m}{2} + 5,\ \frac{3}{2}\gamma + 5,\ \frac{\gamma}{2} + \frac{g-1}{2} + 4\right\}$$

then there is a line bundle $M \in \mathrm{Pic}(S)$ adapted to $|H|$ such that $|A| \subseteq |M \otimes \mathcal{O}_C|$ and $\gamma(M \otimes \mathcal{O}_C) \leq \gamma(A)$. Moreover, one has $c_1(M).C \leq \frac{3g-3}{2}$.

One could hope for a cleaner statement without bounds, however, as we recall in Example 4.20, the Donagi–Morrison conjecture is false for $r \geq 3$. We explain in Remark 6.39 that our bounds are in some sense optimal, at least in a particular case. We state a new bounded version of the Donagi–Morrison conjecture in Section 4.4.

Studying linear systems on curves can also shed light on the geometry of $\mathcal{M}_g$. The Brill–Noether loci stratify $\mathcal{M}_g$ in an intricate way, and the geometry of the Brill–Noether loci is complicated. There are various containments among the Brill–Noether loci, and a question of interest is to determine the maximal loci. This has been crucial in understanding the moduli space of genus 23 curves in work of Eisenbud, Harris, and Farkas on the Kodaira dimension of $\mathcal{M}_{23}$ [24, 26]. Modulo some trivial containments among Brill–Noether loci, we define the *expected maximal Brill–Noether loci* and together with Auel [6], we conjecture that the expected maximal Brill–Noether loci are distinct.

Conjecture 1.3 (Maximal Brill–Noether Loci Conjecture [6])**.** *In every genus g, the maximal Brill–Noether loci are the expected ones, except when $g = 7, 8, 9$.*

Our main result is a verification of the conjecture for certain genus.

Theorem 1.4 ([6])**.** *The maximal Brill–Noether loci conjecture holds in genus $g \leq 19$ and $g = 22, 23$.*

In Section 6.3 we give the proof, explain the exceptional cases of genus 7–9, and outline what remains in genus 20 and 21. We also show how our techniques could be extended by progress on the Donagi–Morrison conjecture.

In the other direction, K3 surfaces can also be studied by understanding the curves they contain. In particular, syzygies of a K3 surface with respect to the morphism induced by the line bundle associated to a curve are in some sense determined by the Clifford index of the curve [1, 81, 82], which builds on a conjecture of Green on the syzygies of canonically embedded curves [33]. The geometry of K3 surfaces, in particular their Picard groups, has been investigated by Knutsen [46, 48, 49, 47, 44]. In particular, the Brill–Noether theory K3 surfaces, as defined by Mukai [62], is closely tied to the Brill–Noether theory of its hyperplane sections. Specifically, a question of interest is whether a polarized K3 surface (S, H) is Brill–Noether special if and only if a curve $C \in |H|$ is Brill–Noether special. We give results concerning Brill–Noether special K3 surfaces using results on the Donagi–Morrison conjecture.

Theorem 1.5. *In genus g with $2 \leq g \leq 19$, a polarized K3 surface (S, H) is Brill–Noether special if and only if a smooth irreducible curve $C \in |H|$ is Brill–Noether special.*

Outline

- In Chapter 2, we give background on lattices, algebraic surfaces, K3 surfaces, the Brill–Noether theory of curves, and stability of sheaves on K3 surfaces.

- In Chapter 3, we give historical background on the role of K3 surfaces in Brill–Noether theory and recall (generalized) Lazarsfeld–Mukai bundles on K3 surfaces, as well as outline Lazarsfeld's proof of the Brill–Noether–Petri theorem. We conclude by giving some useful properties of generalized Lazarsfeld–Mukai bundles.

- In Chapter 4, we summarize Lazarsfeld–Mukai bundle techniques in the work of Green–Lazarsfeld, Donagi–Morrison, and Lelli-Chiesa on the Donagi–Morrison conjecture. We end with a discussion of a new bounded version of the Donagi–Morrison conjecture.

- In Chapter 5, we give our proof of a bounded version of the Donagi–Morrison conjecture for linear systems of rank 3. We study the stability of Lazarsfeld–Mukai bundles of rank 4 and we first reduce the problem to finding a bound on the second Chern class of the Lazarsfeld–Mukai bundle for each terminal filtration. After obtaining bounds for each terminal filtration, we give a proof of our result. This is taken from [6].

- In Chapter 6, we outline the maximal Brill–Noether Loci Conjecture and how results on the Donagi–Morrison conjecture can answer cases of the Maximal Brill–Noether Loci Conjecture. We verify the conjecture in genus $g \leq 19$ and $g = 22, 23$. This is taken from [6].

- In Chapter 7, we present new results on a conjecture of Mukai and Knutsen on the Brill–Noether theory of K3 surfaces. This is joint work with Auel.

Chapter 2

Preliminaries

In this chapter, we provide some background on lattices in Section 2.1, algebraic surfaces in Section 2.2, K3 surfaces in Section 2.3, the Brill–Noether theory of curves in Section 2.4, and the stability of sheaves on K3 surfaces in Section 2.5.

Section 2.1

Lattices

We briefly introduce notation and recall standard definitions concerning integral bilinear forms following [41, Chapter 14].

Definition 2.1. A *lattice* is a free, finite-rank $\mathbb{Z}$-module Λ equipped with a nondegenerate, symmetric, integral bilinear form

$$\Lambda \times \Lambda \to \mathbb{Z}, \quad (a, b) \mapsto (a.b)_\Lambda,$$

which we call the *intersection form* on Λ. We generally write $a.b = (a.b)_\Lambda$ when the lattice Λ is understood. We write $a^2 := a.a$, called the self-intersection of a.

Recall that the intersection form is *nondegenerate* if for every $a \in \Lambda \setminus \{0\}$, there exists $b \in \Lambda$ such that $a.b \neq 0$.

We recall some standard notions for lattices.

- Choosing a basis $\{a_i\}$ for Λ, the *Gram matrix*, or *intersection matrix*, is given by $G_\Lambda = (a_i.a_j)_{i,j}$. The determinant of G_Λ is the *discriminant* of Λ, $\mathrm{disc}(\Lambda)$.

- A lattice Λ is *even* if for every $a \in \Lambda$, we have $a^2 \in 2\mathbb{Z}$.

- The *signature* (n_+, n_-) of Λ is the signature of the natural extension of the intersection form to $\Lambda_\mathbb{R} := \Lambda \otimes_\mathbb{Z} \mathbb{R}$. A lattice is *definite* if either $n_+ = 0$ or $n_- = 0$. Otherwise, it is called *indefinite*.

- The *hyperbolic plane* is the lattice U of rank 2 with intersection matrix given by
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

- A *morphism* between lattices $\varphi : \Lambda_1 \to \Lambda_2$ is a $\mathbb{Z}$-linear map that respects the bilinear forms: $(\varphi(a).\varphi(b))_{\Lambda_2} = (a.b)_{\Lambda_1}$ for all $a, b \in \Lambda_1$. We say that $\varphi : \Lambda_1 \to \Lambda_2$ is of *finite index* if $\varphi(\Lambda_1) \subset \Lambda_2$ has finite index as abelian groups.

- For a lattice Λ, the *dual lattice* is defined by
$$\Lambda^* := \mathrm{Hom}_\mathbb{Z}(\Lambda, \mathbb{Z}),$$
whose Gran matrix is G_Λ^{-1}. Alternatively, one sees that
$$\Lambda^* \cong \{x \in \Lambda_\mathbb{Q} \mid x.b \in \mathbb{Z} \text{ for all } b \in \Lambda \subseteq \Lambda_\mathbb{Q}\},$$
where $\Lambda_\mathbb{Q} := \Lambda \otimes_\mathbb{Z} \mathbb{Q}$.

- There is an injection of finite index

$$i_\Lambda : \Lambda \hookrightarrow \Lambda^*,\ x \mapsto (y \mapsto x.y)\,.$$

  The cokernel of i_Λ is called the *discriminant group* of Λ, denoted

$$A_\Lambda := \Lambda^*/\Lambda.$$

- The discriminant group A_Λ is a finite group of order $|\operatorname{disc}(\Lambda)|$. A lattice is called *unimodular* if $\operatorname{disc}(\Lambda) = \pm 1$.

- The *length* of Λ, denoted $\ell(\Lambda)$, is the minimal number of generators of A_Λ as an abelian group.

- An injective morphism of lattices $\varphi : \Lambda_1 \hookrightarrow \Lambda_2$ is called *primitive* if $\Lambda_2/\varphi(\Lambda_1)$ is torsion free.

- If $\Lambda_1 \hookrightarrow \Lambda_2$ is an embedding such that Λ_2/Λ_1 is a finite group, the *index* of Λ_1 in Λ_2, denoted $[\Lambda_2 : \Lambda_2]$ is the order of Λ_2/Λ_1.

- Let $\Lambda_1 \hookrightarrow \Lambda_2$ be a finite index embedding. From the inclusions

$$\Lambda_1 \hookrightarrow \Lambda_2 \hookrightarrow \Lambda_2^* \hookrightarrow \Lambda_1^*,$$

  it follows that

$$\operatorname{disc}(\Lambda_1) = \operatorname{disc}(\Lambda_2) \cdot (\Lambda_2 : \Lambda_1)^2.$$

- For a lattice Λ and $m \in \mathbb{Z}$, we denote by $\Lambda(m)$ the lattice obtained from Λ by multiplying the intersection form by m: that is, $(a.b)_{\Lambda(m)} := m(a.b)_\Lambda$.

- There is a unique even, unimodular, positive definite lattice of rank 8 up to isomorphism, called the E_8 lattice. The Gram matrix of E_8 is given by

$$\begin{pmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & -1 & & & \\ & & -1 & 2 & 0 & & & \\ & & -1 & 0 & 2 & -1 & & \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & -1 \\ & & & & & & -1 & 2 \end{pmatrix}.$$

- The *K3 lattice* is defined to be $\Lambda_{K3} := U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$.

Remark 2.2. A result of Milnor states that if Λ is an even indefinite unimodular lattice, then $\Lambda \cong U^{\oplus x} \oplus E_8(-1)^{\oplus y}$ for some $x, y \in \mathbb{Z}_{\geq 0}$. Thus the K3 lattice is the unique even unimodular lattice of signature $(3, 19)$.

We recall a theorem of Nikulin [67, Theorem 1.14.4] on the uniqueness of embeddings of lattices.

Theorem 2.3. *Let Λ be an even unimodular lattice of signature (n_+, n_-) and let Λ_1 be an even lattice of signature (m_+, m_-). If $m_+ < n_+$, $m_- < n_-$, and $\ell(\Lambda_1) + 2 \leq n_+ + n_- - m_+ - m_-$, then there exists a primitive embedding $\Lambda_1 \hookrightarrow \Lambda$ which is unique up to post composing by an automorphism of Λ.*

Section 2.2

Algebraic surfaces

We briefly recall facts about algebraic surfaces following [38, 41, 73].

Unless otherwise specified, we let k be an algebraically closed field of characteristic 0. By a *variety* over k, we mean a separated, integral scheme of finite type over k. By a *surface* over k, we mean a smooth algebraic surface, that is, a projective, connected, nonsingular variety over k of dimension 2.

Let S be a surface over k. For a coherent sheaf F on S, if the surface is understood, we write $h^i(F) := h^i(S, F)$. The *Euler characteristic* of F is $\chi(F) = \sum_{i\geq 0} h^i(F)$. We note that

$$\chi(\mathcal{O}_S) = 1 - q + p_g,$$

where $q = h^1(\mathcal{O}_S)$ is the *irregularity* of S and $p_g = h^2(\mathcal{O}_S)$ is the *geometric genus* of S.

Let $\operatorname{Pic}(S)$ be the group of line bundles on S modulo isomorphism. We shall denote linear equivalence by $\sim$ or $=$ if we are only considering divisors up to linear equivalence. The *Néron–Severi* group of S is

$$\operatorname{NS}(S) := \operatorname{Pic}(S)/\operatorname{Pic}^0(S),$$

the quotient of $\operatorname{Pic}(S)$ by the subgroup $\operatorname{Pic}^0(S)$ of line bundles in $\operatorname{Pic}(S)$ algebraically equivalent to $\mathcal{O}_S$.

If the base field k is understood, we write Ω_S instead of $\Omega_{S/k}$ for the sheaf of differentials. We also write K_S or $\omega_S = \det \Omega_S$ for the canonical divisor or canonical line bundle, respectively. For $L \in \operatorname{Pic}(S)$, we denote its dual, $\mathcal{H}om_{\mathcal{O}_S}(\mathcal{O}_S, L)$ by $-L$, L^*, or $L^\vee$. (The three ways of writing the same thing is so we can match other authors who use one of these.)

The Picard group has a symmetric bilinear form $\operatorname{Pic}(S) \times \operatorname{Pic}(S) \to \mathbb{Z}$ given by

$$L_1.L_2 := \chi(\mathcal{O}_S) - \chi(L_1^*) - \chi(L_2^*) + \chi(L_1^* \otimes_{\mathcal{O}_S} L_2^*).$$

Thus $\mathrm{NS}(S)/\mathrm{NS}(S)_{tor}$ is a lattice.

We say a line bundle L is *numerically trivial* if $L.L' = 0$ for all line bundles L', where $L.L'$ denotes the intersection form on $\mathrm{Pic}(S)$. Elements of $\mathrm{Pic}^0(S)$ are numerically trivial. We say L and L' and *numerically equivalent*, denoted by $L \equiv L'$ if $L - L'$ is numerically trivial. Let $\mathrm{Pic}^\tau(S)$ be the subgroup of numerically trivial line bundles. The quotient

$$\mathrm{Num}(S) := \mathrm{Pic}(S)/\mathrm{Pic}^\tau(S)$$

is a quotient of $\mathrm{NS}(S)$. One can also write $\mathrm{Num}(S)$ as the group of Weil divisors on S modulo numerical equivalence.

Proposition 2.4 (Severi's "theorem of the base", Lefschetz $(1,1)$ Theorem)**.** *The groups* $\mathrm{NS}(S)$ *and* $\mathrm{Num}(S)$ *are finitely generated, and* $\rho(S) := \mathrm{rk}\,\mathrm{NS}(S) = \mathrm{rk}\,\mathrm{Num}(S)$.

Definition 2.5. We call $\rho(S)$ the *Picard number* of S.

We recall some facts about the intersection pairing on surfaces.

- If $L_1 = \mathcal{O}_S(C)$ for an integral curve $C \subset S$, then $L_1.L_2 = \deg(L_2|_C)$ if L_1 and L_2 intersect transversely.
- If L is ample, then $L.\mathcal{O}_S(C) > 0$.
- The intersection form restricted to $\mathrm{Num}(S)$ is nondegenerate.

Theorem 2.6 (Hirzebruch–Riemann–Roch)**.** *Let L be a line bundle on S. Then*

$$\chi(L) = \chi(\mathcal{O}_S) + \frac{L.(L - K_S)}{2}.$$

More generally, for a coherent sheaf F on S, one has

$$\chi(F) = \int_S \mathrm{ch}(F).\,\mathrm{td}(S) = [\mathrm{ch}(F).\,\mathrm{td}(S)]_2,$$

where

$$\mathrm{ch}(F) = \mathrm{rk}(F) + c_1(F) + \frac{c_1(F)^2 - 2c_2(F)}{2}$$

is the Chern character of F,

$$\mathrm{td}(S) = 1 - c_1(\Omega_S) + \frac{c_1(\Omega_S)^2 + c_2(\Omega_S)}{12}$$

is the Todd class of Ω_S^*, and $[x]_2$ is the degree 2 piece of x in the Chow ring of S after identifying it with $\mathbb{Z}$.

Corollary 2.7. *For a vector bundle F on S,*

$$\chi(F) = \mathrm{rk}(F) \cdot \chi(\mathcal{O}_S) + \frac{c_1(F).(c_1(F) - K_S)}{2} - c_2(F).$$

Let C be a reduced curve on S. Recall that the *arithmetic genus* of a curve C is $p_a(C) := 1 - \chi(C, \mathcal{O}_C)$.

Theorem 2.8. *Let C be an irreducible curve on S. Then*

$$\omega_C = (\mathcal{O}_S(C) \otimes \omega_S)\,|_C.$$

In particular, one has $2p_a(C) - 2 = C^2 + C.K_S$ for any reduced curve C.

We now recall the Hodge index theorem and useful corollaries. Let H be an ample divisor on S, and let h be the image of H in $\mathrm{Num}(S)_{\mathbb{R}}$. Let

$$h^{\perp} := \{x \in \mathrm{Num}(S)_{\mathbb{R}} \ \mid\ x.h = 0\}.$$

Theorem 2.9 (Hodge index theorem)**.** *The restriction of the intersection form on* $\mathrm{Num}(S)_{\mathbb{R}}$ *is negative definite on* $h^{\perp}$*.*

Since $h^2 > 0$, the intersection form is positive definite on $\langle h \rangle \subseteq \mathrm{Num}(S)_{\mathbb{R}}$. Thus the Hodge index theorem says that the signature of the intersection form on $\mathrm{Num}(S)_{\mathbb{R}}$ is $(1, \rho(S) - 1)$.

We can rephrase this using divisors.

Corollary 2.10. *If D is a divisor such that $D.H < 0$, then either $D^2 < 0$ or $D \equiv 0$.*

Corollary 2.11 ([41, Chapter 2, Remark 2.2])**.** *If L_1 and L_2 are line bundles such that $L_1^2 > 0$, then*

$$L_1^2 \cdot L_2^2 \leq (L_1.L_2)^2.$$

Moreover, if L_2 is not numerically trivial, then equality holds if and only if L_1 and L_2 are linearly dependent in $\mathrm{Num}(S)_{\mathbb{Q}}$*.*

Proof. To prove the first statement, notice that $L_1^2 L_2 - (L_1.L_2)L_1 \in L_1^{\perp}$ and apply the Hodge index theorem.

To prove the second statement, consider the sublattice $\langle L_1, L_2 \rangle \subseteq \mathrm{Num}(S)$ generated by L_1 and L_2. Equality is equivalent to the matrix

$$\begin{pmatrix} L_1^2 & L_1.L_2 \\ L_1.L_2 & L_2^2 \end{pmatrix}$$

having determinant 0, which is equivalent to L_1 and L_2 being dependent over $\mathbb{R}$, in the sense that $\langle L_1, L_2 \rangle_{\mathbb{R}} \subseteq \mathrm{Num}(S)_{\mathbb{R}}$ is 1-dimensional. Since $L_i \in \mathrm{Num}(S)_{\mathbb{Q}} \subseteq \mathrm{Num}(S)_{\mathbb{R}}$, linear dependence over $\mathbb{R}$ actually gives linear dependence over $\mathbb{Q}$. □

Corollary 2.12. *Let L_1 satisfy $L_1^2 > 0$. If $L_2 \in L_1^{\perp}$, then either $L_2^2 < 0$ or L_2 is numerically trivial.*

Proof. By the above, we must have $L_2^2 \leq 0$. If L_2 is not numerically trivial, assume for contradiction that $L_2^2 = 0$. Since we have equality in the corollary above, we have $mL_2 = nL_1$. Furthermore, since $L_1^2 > 0$, we must have $n = 0$, so $mL_2 \equiv 0 \in \mathrm{Num}(S)$. However, since $\mathrm{Num}(S)$ is torsion free, $m = 0$. $\square$

We recall a few facts about divisors on surfaces, some of which we will return to in the following section, when we focus on K3 surfaces. We follow the exposition in [38, 41, 73], other references are [7, 54].

Recall the universal property of projective space.

Proposition 2.13. *Let R be a commutative ring and X a scheme over R. Then the scheme $\mathbb{P}^n_R$ represents the functor $\underline{\mathrm{Sch}}_R \to \underline{\mathrm{Set}}$ given by*

$$X \mapsto \{\textit{rank } 1 \textit{ locally free quotients of } \mathcal{O}_X^{n+1}\}.$$

More concretely, there is a functorial bijection

$$\mathrm{Hom}_R(X, \mathbb{P}^n_R) \leftrightarrow \{\textit{invertible sheaves } L \textit{ on } X \textit{ globally generated by } n+1 \textit{ sections}\}.$$

Thus if L is globally generated, we obtain a morphism $\varphi_{|L|} : S \to \mathbb{P}^{h^0(S,L)-1}$ induced by the global sections of $|L|$. Recall that L is *very ample* if $\varphi_{|L|}$ is an embedding, and L is *ample* if there is a positive integer m such that $\varphi_{|mL|}$ gives an embedding. A divisor D is called (very) ample if the same is true of $\mathcal{O}_S(D)$.

However, if L is not globally generated, we may still obtain a rational map to $\mathbb{P}^n$. To every line bundle L on S, one associates the *complete linear system* $|L| := \mathbb{P}(H^0(S, L))$; equivalently, this is the space of all effective divisors D linearly equivalent to L. The *base locus* $\mathrm{Bs}\,|L|$ of $|L|$ is the maximal closed subscheme of S contained in all effective divisors $D \in |L|$, concretely $\mathrm{Bs}\,|L| = \bigcap_{s \in H^0(S,L)} Z(s)$.

If $h^0(S, L) \geq 2$, then the map induced by $|L|$ on $S \setminus \mathrm{Bs}\,|L|$ gives a rational map $\varphi_{|L|} : S \dashrightarrow \mathbb{P}^{h^0(S,L)-1}$ induced by the global sections of $|L|$.

Since S is a surface, the base locus of L can have components of dimension zero and one. The one dimensional parts of $\mathrm{Bs}\,|L|$ are called the *fixed part*, denoted by F. The natural inclusion $L(-F) \hookrightarrow L$ yields an isomorphism $H^0(S, L) \cong H^0(S, L(-F))$, and we can view $\varphi_{|L|}$ as $\varphi_{|L(-F)|}$. We can extend $\varphi_{|L(-F)|}$ to a morphism $S \setminus \{x_i\} \to \mathbb{P}^n$, where $\{x_i\}$ are the zero dimensional base components of $L(-F)$, which contains the zero dimensional locus of $\mathrm{Bs}\,|L|$. We write $M := L(-F)$, and call it the *mobile part* of L.

We now turn to how properties of line bundles are captured numerically using the intersection form on $\mathrm{NS}(S)$.

Definition 2.14. A line bundle L on S is called *nef* if $L.C \geq 0$ for all closed curves $C \subset S$. L is called *big and nef* if $L^2 > 0$ and L is nef. We do not define the notion of *big* here, we note that it <u>does not</u> mean $L^2 > 0$.

Theorem 2.15 (Kodaira–Ramanujam)**.** *If L is a big and nef line bundle, then $H^i(S, L \otimes \omega_S) = 0$ for $i > 0$.*

Definition 2.16. The *positive cone* $\mathcal{C}_S \subseteq \mathrm{NS}(S)_{\mathbb{R}}$ is the connected component of the set $\{x \in \mathrm{NS}(S)_{\mathbb{R}} \mid x^2 > 0\}$ that contains an ample class. Note that $\mathcal{C}_S$ contains all the ample classes.

The *ample cone* $\mathrm{Amp}(S) \subset \mathrm{NS}(S)_{\mathbb{R}}$ is the set of $\mathbb{R}_{>0}$-linear combinations of ample classes.

The *nef cone* is the set

$$\mathrm{Nef}(S) := \{x \in \mathrm{NS}(S)_{\mathbb{R}} \mid x.C \geq 0 \text{ for all curves } C \subset S\}.$$

Remark 2.17. Note that the positive, ample, and nef cones are all convex cones. It is not in general true that the nef cone is the convex cone spanned by all nef line bundles, though the closure of this does give $\mathrm{Nef}(S)$, see [41, Chapter 8].

The following result relates numerical conditions to the ampleness of line bundles.

Theorem 2.18 (Nakai–Moishezon–Kleiman). *A line bundle L on S is ample if and only if*

$$L^2 > 0 \text{ and } L.C > 0$$

for all curves $C \subset S$.

At least numerically, there are obvious relations among the ample and nef cones, reflected in the following proposition.

Proposition 2.19. $\mathrm{Amp}(S)$ *is open in* $\mathcal{C}_S$ *and* $\mathrm{Nef}(S)$ *is closed in* $\overline{\mathcal{C}_S}$. *Moreover,*

$$\mathrm{Amp}(S) \subset \mathrm{Interior}\,\mathrm{Nef}(S) \subset \mathrm{Nef}(S) = \overline{\mathrm{Amp}(S)}.$$

Corollary 2.20. *For every class x in the boundary $\partial\,\mathrm{Nef}(S)$ of the nef cone, either $x^2 = 0$ or $x.C = 0$ for an irreducible curve $C \subset S$.*

We see then that a divisor D is nef if $D.C \geq 0$ for any irreducible curve $C \subset S$. Equivalently, $D.E \geq 0$ for every effective divisor E on S.

A useful numerical outcome of nef-ness is the following.

Proposition 2.21 (Kleiman). *If L is a nef line bundle, then $L^2 \geq 0$.*

The study of various cones in $\mathrm{Num}(S)_{\mathbb{R}}$ is a vast and rich topic. We will focus on K3 surfaces, and will give further results that can be more explicit once S is assumed to be a K3 surface.

Section 2.3

K3 surfaces

We begin with general notions of K3 surfaces, curves on K3 surfaces, and the moduli of K3 surfaces. We then give more precise cases numerical criteria for ampleness and nefness on K3 surfaces. For those interested in learning more, we recommend [41].

Definition 2.22. A *K3 surface* over k is a complete non-singular variety S of dimension 2 such that

$$\Omega_S^2 \cong \mathcal{O}_S \text{ and } H^1(S, \mathcal{O}_S) = 0.$$

We now assume, unless otherwise specified that $k = \mathbb{C}$.

Example 2.23. We give a few examples of K3 surfaces.

- A smooth quartic $S \subset \mathbb{P}^3$ is a K3 surface, by adjunction and the Lefschetz hyperplane theorem.
- A smooth complete intersection S of type $(d_1, \dots, d_n)$ in $\mathbb{P}^{n+2}$ is a K3 surface if and only if $\sum_{i=1}^n d_i = n + 3$ (as then K_S is trivial, again by adjunction). Under the assumption that all $d_i > 1$, there are in fact only 3 cases: a smooth quartic in $\mathbb{P}^3$, the intersection of a quadric and a cubic in $\mathbb{P}^4$ ($n = 2$ and $d_1 = 2$, $d_2 = 3$), or the intersection of 3 quadrics in $\mathbb{P}^5$ ($n = 3$ and $d_i = 2$). These yield K3 surfaces of degree 4, 6, and 8, respectively. Here by *degree* we mean that the pull back of $\mathcal{O}_{\mathbb{P}^n}(1)$ to S gives a line bundle of the given degree. We give more details on degree when we define polarized K3 surfaces, see Definition 2.24.
- A smooth surface with a degree 2 map $\pi : S \to \mathbb{P}^2$ branched along a smooth sextic curve. This gives a K3 surface of degree 2 (which can be seen by computing ω_S via the Riemann–Hurwitz theorem for surfaces), and is called a *K3 double plane*. This example will be useful in Chapter 4.

Recall the Hodge numbers $h^{p,q} := \dim H^q(S, \Omega_S^p)$, and the Hodge decomposition

$$H^k(S, \mathbb{C}) = \bigoplus_{p+q=k} H^q(S, \Omega_S^p).$$

For a K3 surface, the algebraic classes in $H^2(S, \mathbb{C})$ coincide with Hodge classes $H^{1,1}(S)$: that is, $\mathrm{Pic}(S) \cong H^2(S, \mathbb{Z}) \cap H^{1,1}(S)$. To see this, note that the cycle class map

$$\mathrm{Pic}(S) \to H^2(S, \mathbb{Z}) \cap H^{1,1}(S)$$

is surjective by the Lefschetz $(1,1)$-theorem (the Hodge theorem in codimension 1) for K3 surfaces. The kernel of the cycle class map is $H^1(S, \mathcal{O}_S)/H^1(S, \mathbb{Z})$ by the exponential exact sequence, hence is trivial for K3 surfaces. For more details on Hodge structures, see [41, Chapter 3].

We recall some basic and some nontrivial facts about K3 surfaces.

- $K_S = 0$.
- S has geometric genus $p_g = h^0(S, \omega_S) = 1$.
- $\chi(\mathcal{O}_S) = 2$, hence by Theorem 2.6, $\chi(L) = 2 + \frac{L^2}{2}$.
- For a coherent sheaf F on S, by Theorem 2.6 we have

$$\chi(F) = 2\,\mathrm{rk}(F) + \mathrm{ch}_2(F) = 2\,\mathrm{rk}(F) + \frac{c_1(F).(c_1(F) - K_S)}{2} - c_2(F).$$

- There are isomorphisms $\mathrm{Pic}(S) \cong \mathrm{NS}(S) \cong \mathrm{Num}(S)$.
- $\mathrm{Pic}(S)$ is torsion free. Indeed, if L were a torsion line bundle, then $L^2 = 0$ so we have $\chi(L) = 2$, hence L of $-L$ is effective. But a nonzero section of $\pm L$ would give a nonzero section of $\pm mL$. Thus if mL is trivial, so is L.

- The algebraic fundamental group $\pi_1(S)$ is trivial, which follows from the fact that $H^1(S, \mathbb{Z}) = 0$.

- The intersection form on $\mathrm{Pic}(S)$ is even, non-degenerate, and of signature $(1, \rho - 1)$.

- $h^1(S, \omega_S) = h^1(S, \mathcal{O}_S) = 0$, which is crucial in the relation between K3 surfaces and curves.

- $h^0(S, \omega_S) = h^2(S, \mathcal{O}_S) = 1$, i.e. $H^{2,0}(S) = H^0(S, \Omega_S^2) = k\omega_S$.

- By the Hodge decomposition and Theorem 2.6, we have $2h^0(S, \Omega_S) - h^1(S, \Omega_S) = \mathrm{ch}_2(\Omega_S) + 4 = 4 - c_2(\Omega_S) = -20$. Since $h^0(S, \Omega_S) = 0$, we have $h^1(S, \Omega_S) = 20$.

- The Hodge diamond

$$
\begin{array}{ccccc}
 & & h^{0,0} & & \\
 & h^{1,0} & & h^{0,1} & \\
h^{2,0} & & h^{1,1} & & h^{0,2} \\
 & h^{2,1} & & h^{1,2} & \\
 & & h^{2,2} & &
\end{array}
$$

of a K3 surface is

$$
\begin{array}{ccccc}
 & & 1 & & \\
 & 0 & & 0 & \\
1 & & 20 & & 1 \\
 & 0 & & 0 & \\
 & & 1 & &
\end{array}
$$

- $H^2(S, \mathbb{Z})$ together with the cup product is an even unimodular lattice of signature $(3, 19)$, hence isomorphic to the K3 lattice Λ_{K3}, see Remark 2.2.
- Since $\mathrm{Pic}(S) = H^2(S, \mathbb{Z}) \cap H^{1,1}(S)$, we have $\rho(S) \leq \dim_{\mathbb{C}} H^{1,1}(S) = 20$.

Definition 2.24. By a *polarized K3 surface* (S, H) we mean a K3 surface S together with an ample line bundle H. We say (S, H) is of *degree* d if $H^2 = d$.

Note that for any line bundle L, the self-intersection L^2 is even, and thus we can always write $H^2 = d = 2g - 2$. Because the arithmetic genus of any smooth curve in $|H|$ is g.

Definition 2.25. We say (S, H) is a *polarized K3 surface of genus* g when $H^2 = 2g-2$.

Remark 2.26. There is a more general notion of *lattice polarized* K3 surfaces, defined as follows. Let Λ be an even nondegenerate lattice of signature $(1, \rho-1)$, and let H be a distinguished class positive norm. Then a Λ-*polarized K3 surface* is a polarized K3 surface (S, H) together with a primitive isometric embedding $\Lambda \hookrightarrow \mathrm{Pic}(S)$ preserving H. For more on the moduli of lattice polarized K3 surfaces, see [20].

The moduli space of K3 surfaces becomes better behaved once one chooses a polarization. In particular, the moduli space of K3 surfaces of degree d is a quasiprojective variety of dimension 19. Moreover, there is a Torelli-type theorem for algebraic

K3 surfaces due to Bogomolov, Kulivok, Piatetskii-Shapiro, Shafarevich, and Tjurin [11, 10, 50, 72, 76, 83] which we briefly recall following notes by Várilly-Alvarado [80, Section 1.9].

Definition 2.27. A *marking* on a complex K3 surface S is an isometry of lattices $\Phi : H^2(S,\mathbb{Z}) \to \Lambda_{K3}$. By a *marked K3 surface*, we mean a pair (S,Φ). We can extend Φ to a map $\Phi_{\mathbb{C}} : H^2(S,\mathbb{C}) \to \Lambda_{K3} \otimes \mathbb{C}$.

Let

$$\Omega := \{x \in \mathbb{P}(\Lambda_{K3} \otimes \mathbb{C}) \mid (x.x) = 0, (x.\overline{x}) > 0\},$$

where $(,)$ denotes the intersection form on $\Lambda_{K3} \otimes \mathbb{C}$. We call Ω the *period domain of complex K3 surfaces.*

We call $\Phi_{\mathbb{C}}(\mathbb{C}\omega_S) \in \mathbb{P}(\Lambda_{K3} \otimes \mathbb{C})$ the *period point of* (S,Φ).

It is easy to see that Ω is an open set (in the complex topology) of a quadric in $\mathbb{P}(\Lambda_{K3} \otimes \mathbb{C})$.

Theorem 2.28 (Weak Torelli Theorem)**.** *Two complex K3 surfaces S and S' are isomorphic if and only if there are markings*

$$\begin{tikzcd} H^2(S,\mathbb{Z}) \arrow[dr, "\Phi"] & \\ & \Lambda_{K3} \\ H^2(S',\mathbb{Z}) \arrow[ur, "\Phi'"'] & \end{tikzcd}$$

whose period points in Ω coincide.

Theorem 2.29 (Strong Torelli Theorem)**.** *Let (S,Φ) and (S',Φ') be marked complex K3 surfaces whose period points in Ω coincide. Suppose that*

$$f^* = (\Phi')^{-1} \circ \Phi : H^2(S,\mathbb{Z}) \to H^2(S',\mathbb{Z})$$

satisfies $f^*(\mathcal{C}_S) \subseteq \mathcal{C}_{S'}$ *and induces a bijection on effective classes. Then there is a unique isomorphism* $f : S' \to S$ *inducing* f^**.*

We would like to deal more explicitly with the Picard lattices of K3 surfaces, hence we recall the key theorems which tell us that the geometry of a K3 surface is in some way determined by its Picard lattice. We follow [80, Section 1.10]. Ideally, we would like a K3 surface to be determined by its Picard lattice, which is not the case. So we would like to at least know that there is a K3 surface with a particular lattice we are interested in. This is true; in fact, given a Picard group of rank ρ, there is a $20-\rho$ dimensional space of K3 surfaces with that Picard group. In particular, a very general algebraic K3 surface has Picard rank 1, i.e., $\mathrm{Pic}(S) = \mathbb{Z}[H]$.

By definition, a point $x \in \mathbb{P}(\Lambda_{K3} \otimes \mathbb{C})$ gives a 1-dimensional subspace in $\Lambda_{K3} \otimes \mathbb{C}$. From x, we want to construct a Hodge decomposition

$$H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$$

of $\Lambda_{K3} \otimes \mathbb{C}$ that would be the Hodge decomposition if we started with a K3 surface. Let the 1-dimensional subspace associated to x be $H^{2,0} \subset \Lambda_{K3} \otimes \mathbb{C}$, let $H^{0,2} = \overline{H^{2,0}}$ be the conjugate linear subspace, and let $H^{1,1}$ be the orthogonal complement of $H^{2,0} \oplus H^{0,2}$ in $\Lambda_{K3} \otimes \mathbb{C}$ with the induced bilinear form. We say that

$$H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$$

is a *decomposition of K3 type* for $\Lambda_{K3} \otimes \mathbb{C}$.

Theorem 2.30 (Surjectivity of the period map)**.** *Let* $x \in \Omega$ *be a point inducing a decomposition*

$$\Lambda_{K3} \otimes \mathbb{C} = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$$

of K3 type, then there exists a complex K3 surface S and a marking $\Phi : H^2(S, \mathbb{Z}) \to \Lambda_{K3}$ such that $\Phi_{\mathbb{C}}$ preserves the Hodge decomposition.

Recall that $\mathrm{Pic}(S) = H^2(S, \mathbb{Z}) \cap H^{1,1}(S)$. Thus we are interested in which lattices we can embed into $H^2(S, \mathbb{Z}) \cong \Lambda_{K3}$.

Theorem 2.31 ([67, Corollary 1.12.3])**.** *There exists a primitive embedding $\Lambda \hookrightarrow \Lambda_{K3}$ of an even lattice Λ of rank r and signature $(p, r-p)$ into Λ_{K3} if $p \leq 3$, $r - p \leq 19$, and $\ell(\Lambda) \leq 22 - r$.*

In particular, if Λ is an even lattice of rank 2 whose Gram matrix has negative discriminant, then there is a K3 surface with $\mathrm{Pic}(S) = \Lambda$.

We conclude this section by recalling that the moduli spaces of polarized K3 surfaces are nice.

Theorem 2.32. *For each $d > 0$, the moduli space of polarized complex K3 surfaces of degree $2d$ is an irreducible quasi-projective variety of dimension 19.*

Let (S, H) be a polarized K3 surface of genus g. In the moduli space $\mathcal{K}_g^\circ$ of polarized K3 surfaces of genus g, the Noether–Lefschetz (NL) locus parameterizes K3 surfaces with Picard rank > 1. We will mostly be interested in K3 surfaces of Picard rank 2. In particular, we define the rank 2 lattice $\Lambda_{g,d}^r$ to be the lattice with intersection matrix

$$\Lambda_{g,d}^r := \begin{array}{c|cc} & H & L \\ \hline H & 2g-2 & d \\ L & d & 2r-2 \end{array}.$$

By Hodge theory, the NL locus is a union of countably many irreducible divisors, which we call NL divisors. We define the NL divisor $\mathcal{K}_{g,d}^r$ to be the locus of polarized K3 surfaces $(S, H) \in \mathcal{K}_g$ such that admits a primitive embedding of $\Lambda_{g,d}^r$ in $\mathrm{Pic}(S)$ preserving H. We note that the $\mathcal{K}_{g,d}^r$ are each irreducible by [68].

2.3.1. Curves on K3 surfaces

We now turn to the main focus of this thesis, curves on K3 surfaces. We briefly recall some basic facts about curves on K3 surfaces, and then return to numerical criteria for ampleness and other properties of line bundles on K3 surfaces.

Proposition 2.33. *Let $C \subset S$ be an irreducible curve. Then the following statements hold.*

(i) $\omega_C = \mathcal{O}_S(C)|_C$.

(ii) $C^2 = 2p_a(C) - 2$ *hence* $C^2 \geq -2$.

(iii) $C^2 = -2$ *if and only if* C *is a smooth rational curve.*

(iv) $C^2 = 0$ *if and only if* $p_a(C) = 1$.

(v) *If* $p_a(C) \geq 2$*, then* C *is big and nef.*

Proof. Since S is a K3 surface, we have $K_S = 0$, thus the adjunction formula reads

$$\omega_C \cong \mathcal{O}_S(C)|_C,$$

and taking degrees gives $2p_a(C) - 2 = C^2$. This proves (i)-(iv). To prove (v), we note that if $p_a(C) \geq 2$, then $C^2 > 0$ and for any other irreducible curve $C' \subset S$, $C.C' > 0$ as C' cannot be a component of the irreducible curve C. Thus C is big and nef if $p_a(C) \geq 2$. □

Definition 2.34. By a (-2)-*curve* on a K3 surface, we mean an irreducible curve C with $C^2 = -2$.

Note that a (-2)-curve is in fact integral. Thus for a (-2)-curve C, the above implies that $p_a(C) = 0$, and $p_g(C) = 0$, hence C is smooth. Therefore C is rational.

We recall a few consequences of the Riemann–Roch theorem on K3 surfaces.

Proposition 2.35 ([41, Chapter 2, Section 1.4]). *Let L be a line bundle on S.*

(i) If $L^2 \geq -2$, then $H^0(S, L) \neq 0$ or $H^0(S, -L) \neq 0$. The converse does not hold.

(ii) If $L^2 \geq 0$, then either $L \cong \mathcal{O}_S$ or $h^0(S, L) \geq 2$ or $h^0(S, -L) \geq 2$.

(iii) If $h^0(S, L) = 1$ and $D \subset S$ is the effective divisor defined by the unique section of L, then every curve $C \subseteq D$ satisfies $C^2 \leq -2$, and if C is integral, then $C^2 = -2$, whereby C is rational.

Proof. Straightforward computations with the Riemann–Roch theorem prove (i) and (ii). To prove (iii), write $D = \sum a_i C_i$. Since $h^0(S, D) = 1$, we have $h^0(S, C_i) = 1$ for any integral component C_i of D. Thus Riemann–Roch gives $C_i^2 = -2$. □

Corollary 2.36 ([41, Chapter 2, Corollary 1.3]). *The fixed part F of a line bundle L on a K3 surface is a linear combination of smooth rational curves, i.e., $F = \sum a_i C_i$ with $a_i \geq 0$ and C_i rational.*

Proof. Write $L = M + F$, where F is the fixed part of L and M is the mobile part of L. Then $h^0(S, F) = 1$ since F is fixed. The result follows from (iii) of the previous proposition. □

Using this, we can improve the Nakai–Moishezon–Kleiman characterization of the ample cone, Theorem 2.18, in the case of K3 surfaces.

Proposition 2.37. *The closure of the ample cone* $\mathrm{Amp}(S)$ *is*

$$\mathrm{Nef}(S) = \{x \in \mathcal{C}_S \mid x.C \geq 0 \textit{ for all } (-2)\textit{-curves } C\}.$$

That is, a line bundle L on S is ample if and only if $L.C > 0$ for every smooth rational curve $C \subset S$.

Corollary 2.38. *Let L be a line bundle on S satisfying $L^2 \geq 0$ and $L.C \geq 0$ for all smooth rational curves $C \subset S$. Then L is nef unless S has no smooth rational curves, in which case L or $-L$ is nef.*

In summary, the ampleness and nefness of a line bundle can be checked numerically.

Proposition 2.39. *Let L be a line bundle on a K3 surface S.*

(i) L is ample if and only if $L \in \mathcal{C}_S$ (L is in the positive cone of X) and $L.C > 0$ for all smooth rational curves C on S.

(ii) If L is effective and $L^2 \geq 0$, then L is nef if and only if $L.C \geq 0$ for all smooth rational curves C on S.

We now look more closely at smooth curves on K3 surfaces, and state an important result of Saint-Donat.

Recall that if $C \subset S$ is a smooth curve of genus g, then $C^2 = 2g - 2$. Moreover, from the short exact sequence for the divisor C, we have

$$0 \to H^0(S, \mathcal{O}_S) \to H^0(S, \mathcal{O}_S(C)) \to H^0(C, \mathcal{O}_S(C)|_C) \to H^1(S, \mathcal{O}_S) = 0,$$

whereby $h^0(S, \mathcal{O}_S(C)) = g+1$. Clearly, since C is effective, $h^2(S, C) = h^0(S, -C) = 0$. Moreover, since $\chi(\mathcal{O}_S(S)) = g + 1$, we have $h^1(S, \mathcal{O}_S(C)) = 0$.

For an irreducible curve C with $C^2 > 0$, we have that $\mathcal{O}_S(C)$ is big and nef. We've just seen that $h^1(S, \mathcal{O}_S(C)) = 0$ as well. In fact this holds for any big and nef line bundle on S.

Proposition 2.40 ([41, Chapter 2, Proposition 3.1])**.** *Let L be a big and nef line bundle on S. Then $H^1(S, L) = 0$.*

Proof. Since L is big and nef, Riemann–Roch gives $h^0(S, L) \geq 3$. Thus there is some (potentially reducible) curve $C \in |L|$, and $L = \mathcal{O}_S(C)$. We want to show that C is integral, in which case the result would follow from the discussion above.

Pick a maximal integral component $C_1 \subset C$ with $h^0(C_1, \mathcal{O}_{C_1}) = 1$. We claim $C = C_1$. If not, then $C_1.(C - C_1) \geq 1$, since C is 1-connected [41, Chapter 2, Remark 1.7]. Thus there is an integral component C_2 of $C - C_1$ such that $C_1.C_2 \geq 1$, hence $H^0(S, \mathcal{O}_{C_2}(-C_1)) = 0$. Thus considering the short exact sequence

$$0 \to H^0(S, \mathcal{O}_{C_2}(-C_1)) \to H^0(S, \mathcal{O}_{C_1+C_2}) \to H^0(S, \mathcal{O}_{C_1}),$$

shows that $C_1 + C_2$ also satisfies $h^0(S, \mathcal{O}_{C_1+C_2}) = 1$, contradicting the maximality of C_1. Hence $L = \mathcal{O}_S(C)$ for an integral curve, and the result follows. □

We summarize a classical result of Saint-Donat from [77].

Theorem 2.41 ([57, Theorem 1.5])**.** *Let L be a line bundle on a K3 surface S such that $h^0(S, L) > 0$. Then, $|L|$ has no base points outside its fixed components. Moreover, if $|L|$ has no basepoints, then either*

(i) $L^2 > 0$, and $h^1(S, L) = 0$ and a general element in $|L|$ is a smooth, irreducible curve of genus $g = 1 + \frac{L^2}{2}$; or,

(ii) $L^2 = 0$, and $L = \mathcal{O}_S(rE)$ for an irreducible curve E with $p_a(E) = 1$ and $r \geq 1$. In this case, one has $h^0(S, L) = r + 1$, $h^1(S, L) = r - 1$ and every element in $|L|$ can be written as a sum $E_1 + \cdots + E_r$ with $E_i \in |E|$.

Section 2.4

Brill–Noether theory for curves

In this section, we recall facts about the Brill–Noether theory of curves, and highlight important theorems in the subject. For an introduction to Brill–Noether theory, we highly recommend reading [35] for an overview of classical Brill–Noether theory and [4] for detailed constructions and proofs.

In brief, "Brill–Noether theory" is the study of how a curve C of genus g can be embedded in projective space $\mathbb{P}^n$. Specifically, by Proposition 2.13, this is equivalent to studying *linear systems* on C. One is then interested in many kinds of questions including: Does C admit a map of degree d to $\mathbb{P}^n$? If so, how many? If not, how *special* is the curve?

Throughout this section, let C be a smooth algebraic curve of genus g.

Definition 2.42. A *linear system of type* g^r_d on C is a pair (A, V) of a line bundle A of degree d on C and V a $(r+1)$-dimensional subspace of $H^0(C, A)$. When $V = H^0(S, A)$, we say that the linear system is *complete*.

We see that maps $C \to \mathbb{P}^r$ of degree d correspond to linear systems (A, V) of type g^r_d with no basepoints.

Definition 2.43. Given a line bundle A on C, we say A has *rank* r if $r = \mathrm{rk}(A) := h^0(C, A) - 1$, and A has *degree* d if $A \in \mathrm{Pic}^d(C)$. We say that A is a line bundle of *type* g^r_d if $\mathrm{rk}(A) = r$ and $\deg(A) = d$, i.e. the complete linear system of A is a g^r_d as above.

Theorem 2.44. *Let A be a line bundle of type g^r_d on a curve C of genus g. Then*

$$\chi(C, L) = h^0(C, A) - h^1(C, A) = r + 1 - h^1(C, A) = d - g + 1.$$

If F is a vector bundle on a curve of genus g, then

$$\chi(C, F) = h^0(C, F) - h^1(C, F) = c_1(F) + \mathrm{rk}(F)(1 - g),$$

where $c_1(F) = \deg(F)$ is the degree of the line bundle $\det(F)$ on C.

Definition 2.45. A line bundle A on C is called *special* if $h^1(C, A) > 0$.

We denote by $\mathcal{M}_g$ the coarse moduli space of smooth curves of genus $g \geq 2$. Deligne and Mumford proved in [19] that $\mathcal{M}_g$ is an irreducible, quasi-projective variety of dimension $3g-3$ over k. We shall say that a property of a curve is an *open property* if it holds for curves inside a nonempty (Zariski) open set of $\mathcal{M}_g$.By a *general curve*, or a property holding on a general curve, we mean that there is some nonempty (Zariski) open set of $\mathcal{M}_g$ of such curves.

We now turn to the central objects of study in Brill–Noether theory.

Definition 2.46. Let C be a curve. The space

$$G^r_d(C) := \{(A, V) \mid (A, V) \text{ is a } g^r_d \text{ on } C\}$$

parameterizes the linear systems on C of degree d and rank r.

The space $W^r_d(C) := \{A \in \mathrm{Pic}^d(C) \mid \mathrm{rk}(A) \geq r\}$ is the image of $G^r_d(C)$ in $\mathrm{Pic}(C)$ under tha natural map

$$\begin{aligned} G^r_d(C) &\to \mathrm{Pic}(C) \\ (A, V) &\mapsto A \end{aligned}$$

.

For details on how the schemes $G^r_d(C)$ can be defined and constructed, see [4].

These spaces also admit global versions, namely moduli spaces $\mathcal{G}^r_{g,d}$ and $\mathcal{W}^r_{g,d}$ with maps to $\mathcal{M}_g$. That is,

$$\mathcal{G}^r_{g,d} := \{(C,(A,V)) \mid [C] \in \mathcal{M}_g, (A,V) \in G^r_d(C)\}$$

which has a map to $\mathcal{M}_g$ with fiber $G^r_d(C)$ over $[C] \in \mathcal{M}_g$. Similarly, we define

$$\mathcal{W}^r_{g,d} := \{(C,A) \mid C \in \mathcal{M}_g, A \in W^r_d(C)\}$$

with a similar map to $\mathcal{M}_g$. In Sections 6.2 and 6.3, we focus on the *Brill–Noether loci* $\mathcal{M}^r_{g,d}$, namely the image of $\mathcal{G}^r_{g,d} \to \mathcal{M}_g$ for some r and d. The Brill–Noether loci can also be defined as the degeneracy loci of maps of vector bundles on $\mathcal{M}_g$ [79], and in the cases we consider are a proper subvariety of $\mathcal{M}_g$.

Results in Brill–Noether theory concern the dimensions and geometric properties of these spaces. Namely, the Brill–Noether theorem answers the questions of how many maps of degree d to $\mathbb{P}^n$ curves admit.

Definition 2.47. Let g, r, d be positive integers. The *Brill–Noether number* is

$$\rho(g,r,d) := g - (r+1)(g-d+r).$$

If A is a line bundle of type g^r_d on a curve C of genus g, we write $\rho(C,A)$, or $\rho(A)$, if C is understood for $\rho(g,r,d)$.

Remark 2.48. For a line bundle A of type g^r_d on a curve of genus g, by Riemann–Roch (Theorem 2.44) we have

$$\rho(C,A) = g - h^0(C,A) \cdot h^1(C,A).$$

We now state various theorems that are sometimes all together known as "The Brill–Noether Theorem", parts of which were proved by Gieseker, Grittiths, Harris, Fulton, Kempf, and Lazarsfeld.

Theorem 2.49 (Brill–Noether Theorem)**.** *Let g, r, d be positive integers. Then the following statements hold.*

(i) *If $\rho(g,r,d) \geq 0$, then $W^r_d(C) \neq \varnothing$ for all $[C] \in \mathcal{M}_g$. Furthermore, if $g-d+r \geq 0$, then each irreducible component of $W^r_d(C)$ has dimension at least $\rho(g,r,d)$.*

(ii) *If $\rho(g,r,d) \geq 0$, then $G^r_d(C) \neq \varnothing$ for any $C \in \mathcal{M}_g$.*

(iii) *If $\rho(g,r,d) < 0$, a general curve in $\mathcal{M}_g$ admits* no *g^r_d.*

(iv) *For a general curve $C \in \mathcal{M}_g$, $\dim G^r_d(C) = \rho(g,r,d)$ and the variety $G^r_d(C)$ is smooth.*

(v) *For a general curve $C \in \mathcal{M}_g$, if $\rho(g,r,d) > 0$, then the variety $G^r_d(C)$ is irreducible.*

(vi) *$\mathcal{G}^r_{g,d}$ has a unique irreducible component that surjects onto $\mathcal{M}_g$ if and only if $\rho(g,r,d) \geq 0$, and that component has relative dimension exactly $\rho(g,r,d)$.*

(vii) *Specifically, when $\rho(g,r,d) = 0$, a general curve $C \in \mathcal{M}_g$ has a finite number of g^r_d's, given by*

$$\#W^r_d(C) = g! \prod_{i=0}^{r} \frac{i!}{(g-d+r+i)!}.$$

(viii) *For $C \in \mathcal{M}_g$ general, and any $(A,V) \in G^r_d(C)$, the Petri map (given by multiplication of sections)*

$$\mu_0 : H^0(C,A) \otimes H^0(C,\omega_C \otimes A^*) \to H^0(C,\omega_C)$$

is injective.

For the proofs and additional context, see [4, 35, 36].

We elaborate on (*viii*) following the exposition in [73].

For a line bundle A of type g^r_d on C, there is a multiplication map

$$\mu_0 : H^0(C, A) \otimes H^0(C, \omega_C \otimes A^*) \to H^0(C, \omega_C),$$

which is called the *Petri map* associated to A on C. Note that the domain of μ_0 has dimension $(r+1)\cdot(g-d+r)$, while the target has dimension g. Therefore the image of μ_0 has codimension at least $\rho(g, r, d)$, potentially higher if μ_0 is not injective. In particular, μ_0 is injective if and only if $\operatorname{im}\mu_0$ has codimension exactly $\rho(g, r, d)$. Dualizing μ_0 and applying Serre duality gives

$$\mu_0^\vee : H^1(C, \mathcal{O}_C) \to H^0(C, A)^\vee \otimes H^1(C, A).$$

Thus μ_0 is injective if and only if $\dim\ker\mu_0^\vee = \rho(g, r, d)$, as the kernel of $\mu_0^\vee$ is the cokernel of μ_0.

The geometric properties of $W^r_d(C)$ are closely related to μ_0.

Proposition 2.50 ([4, Chapter 4, Proposition 4.2])**.** *Let $A \in W^r_d(C) \setminus W^{r+1}_d(C)$. Then the following hold.*

(i) *The tangent space to $W^r_d(C)$ at A is isomorphic to* $\ker\mu_0^\vee$.

(ii) *The tangent space to $W^r_d(C)$ at a point $B \in W^{r+1}_d(C)$ is the whole tangent space $T_B\operatorname{Pic}^d(C)$. In particular, if $W^r_d(C)$ has dimension $\rho(g, r, d)$ and $g - d + r > 0$, then B is a singular point of $W^r_d(C)$.*

Therefore, for $A \in W^r_d(C) \setminus W^{r+1}_d(C)$, we have

$$\rho(g,r,d) \leq \dim_A W^r_d(C) \leq \dim \ker \mu_0^\vee.$$

Together with the discussion of the injectivity of μ_0, we have the following.

Corollary 2.51. *$W^r_d(C)$ is smooth of dimension $\rho(g,r,d)$ at a point $A \in W^r_d(C) \setminus W^{r+1}_d(C)$ if and only if μ_0 is injective.*

Petri conjectured that for a general curve C in $\mathcal{M}_g$, all line bundles have injective Petri map μ_0. This is an open condition in $\mathcal{M}_g$, hence it suffices to find a single example (for each genus) of a curve satisfying Petri's condition, namely that all Petri maps on C are injective. An explicit example was recently constructed in [3]. The conjecture that the Petri map is injective for general curves was proven by Gieseker.

Theorem 2.52 (Brill–Noether–Petri Theorem, [31])**.** *For a general curve C, all line bundles on C have injective Petri map μ_0.*

Gieseker's proof uses a degeneration argument to stable curves, we do not give details of this proof. A different proof was found later by Lazarsfeld [53]. In particular, Lazarsfeld's proof uses K3 surfaces to find smooth curves which satisfy Petri's condition. Our results, and many other results in Brill–Noether theory of curves and K3 surfaces, use the techniques developed by Lazarsfeld, so we recall the proof after introducing the relevant objects, see Chapter 3.

The question "If C does admit a g^r_d, but $\rho(g,r,d) < 0$, how *special* is C?" is still left unanswered. That is, one would like to know if $\rho(g,r,d) < 0$, how many curves have a g^r_d, or what constraints exist on curves with a g^r_d. A coarse answer would be to give the codimension of the locus of such curves. In general, this is quite difficult, though there has been by progress by Cook-Powell–Jensen, Edidin, Eisenbud–Harris, Pflueger, and others [14, 14, 12, 17, 22, 24, 45, 71].

Definition 2.53. A line bundle A of type g^r_d on a curve C of genus g is called *Brill–Noether special* if $\rho(C, A) < 0$. A curve C is called Brill–Noether special if it admits a Brill–Noether special line bundle. When $\rho(g, r, d) < 0$, the *Brill–Noether locus* $\mathcal{M}^r_{g,d}$ is the subvariety of $\mathcal{M}_g$ parameterizing curves admitting a line bundle of type g^r_d.

While the Brill–Noether locus $\mathcal{M}^r_{g,d}$ has expected codimension $-\rho$ in $\mathcal{M}_g$, the geometry of Brill–Noether loci is complicated by the existence of multiple components with some that may be non-reduced and not of the expected dimension. The codimension of components of $\mathcal{M}^r_{g,d}$ is bounded above by $-\rho$ when $\rho < 0$, see [28], but the actual codimensions could be lower and known examples exist when $-\rho > g - 3$, see [71]. On the other hand, when $\rho(g, r, d) = -1$, Eisenbud and Harris [24] show that $\mathcal{M}^r_{g,d}$ is irreducible of codimension 1. More generally, when $-3 \leq \rho \leq -1$, any component of $\mathcal{M}^r_{g,d}$ has codimension $-\rho$, see [22, 24, 78].

A question of interest is then to determine the stratification of $\mathcal{M}_g$ by Brill–Noether loci and, in particular, to identify those loci that are maximal with respect to containment. For Brill–Noether divisors, this is equivalent to having distinct support, a property that is crucially used by Eisenbud and Harris [23], and further developed by Farkas [26], to give lower bounds on the Kodaira dimension of $\mathcal{M}_{23}$. We will return to the question of distinguishing Brill–Noether loci and identifying maximal loci in Chapter 6.

We conclude this chapter by recalling two further important notions which capture how "special" a curve is.

Definition 2.54. The *gonality* $\mathrm{gon}(C)$ of a curve C is

$$\mathrm{gon}(C) := \min\{d \mid C \text{ has a } g^1_d\}.$$

The gonality measures of how far C is from being rational. Indeed, if $\mathrm{gon}(C) = 1$, then C is rational. The idea of how far a curve is from being rational has been extended to varieties and has become a quickly growing topic of research, going under the name "degree of irrationality".

A curve of genus ≥ 2 is called *hyperelliptic* if C admits a finite map $C \to \mathbb{P}^1$ of degree 2. In particular, C is hyperelliptic if and only if $\mathrm{gon}(C) = 2$.

By the Brill–Noether theorem, we see that the gonality of curves can depend on the genus. In particular, the general curve of genus g has g^1_d if and only if $\rho(g, 1, d) = 2d - g - 3 \geq 0$. Solving for d, we see that a general curve of genus g admits a g^1_d if and only if $d \geq \lfloor \frac{g+3}{2} \rfloor$. In particular, every curve of genus g has a $g^1_{\lfloor \frac{g+3}{2} \rfloor}$ by Theorem 2.49(ii), hence $\mathrm{gon}(C) \leq \lfloor \frac{g+3}{2} \rfloor$, with equality for a general curve in $\mathcal{M}_g$. We call $\lfloor \frac{g+3}{2} \rfloor$ the *maximal* or *general* gonality of a curve of genus g.

The study of the Brill–Noether theory of curves of a fixed gonality has been very fruitful and uses many techniques from degeneration to chains of elliptic curves to tropical geometry, see [16, 27, 43, 51, 52, 69, 70].

Generalizing this to higher rank, one can prove a more general result.

Lemma 2.55. *For $x \in \mathbb{R}_{>0}$, we have*

$$\lceil x \rceil - 1 = \begin{cases} \lfloor x \rfloor & x \notin \mathbb{Z}_{>0} \\ x - 1 & x \in \mathbb{Z}_{>0} \end{cases}.$$

Thus $\lceil x \rceil - 1$ is the smallest integer strictly less than x.

Proposition 2.56. *Let g, r, d be positive integers. For fixed g and fixed r, the maximal d such that $\rho(g, r, d) < 0$ is $d = \lceil r + \frac{rg}{r+1} \rceil - 1$.*

Proof. Simply rearranging $\rho(g, r, d) = g - (r+1)(g - d + r) < 0$ gives $d < r + \frac{rg}{r+1}$. $\square$

Another invariant for curves which we focus on is the Clifford index.

Definition 2.57. The *Clifford index* of a line bundle A on C is the integer

$$\gamma(A) = \deg(A) - 2\,\mathrm{rk}(A)$$

where $r(A) = h^0(C, A) - 1$ is the rank of A.

We say that a line bundle A on C *contributes to the Clifford index* of C if $h^0(C, A) \geq 2$ and $h^1(C, A) \geq 2$.

The *Clifford index* of C is defined by

$$\gamma(C) := \min\{\gamma(A) \mid h^0(C, A) \geq 2 \text{ and } h^1(C, A) \geq 2\}.$$

We say that a line bundle A on C *computes* the Clifford index of C if $\gamma(A) = \gamma(C)$.

Remark 2.58. In the definition of the Clifford index, we assume $h^1(C, A) \geq 2$ to the trivial and canonical line bundle, which always have $h^1 = 1$ and Clifford index 0.

Note that by the Riemann–Roch theorem, we have $\gamma(A) = \gamma(\omega_C \otimes A^*)$. In particular, if A contributes to the Clifford index, then $\deg(A) \leq 2g - 2$.

Remark 2.59. If C is hyperelliptic, then $\gamma(C) = 0$, and the converse follows from Clifford's theorem, below.

Clifford's theorem, below, is in some sense a classical theorem in Brill–Noether theory. While it does not directly say anything about varieties we have introduced, it does give bounds on what line bundles curves can carry.

Theorem 2.60 (Clifford's Theorem)**.** *If A is an effective line bundle of degree d on C with $d < 2g - 2$, then $r(A) \leq \frac{d}{2}$. Moreover, if $r(A) = \frac{d}{2}$, then either L is trivial, L is the canonical line bundle, or C is hyperelliptic and L is a multiple of a g^1_2 on C.*

Thus we have $\gamma(C) \geq 0$, with equality if and only if C is hyperelliptic. Similarly, one can prove that if $\gamma(C) = 1$, then C is either trigonal or a plane cubic. Moreover, since a g^1_d has Clifford index $d-2$, $\gamma(C) \leq \text{gon}(C) - 2$, we immediately obtain

$$\gamma(C) \leq \left\lfloor \frac{g-1}{2} \right\rfloor,$$

and by the Brill–Noether theorem equality holds for general curves in $\mathcal{M}_g$. Hence we call $\lfloor \frac{g-1}{2} \rfloor$ the *general Clifford index.*

There is a relation between the gonality and Clifford index of a curve. Namely, one has, as in [18],

$$\gamma(C) + 2 \leq \text{gon}(C) \leq \gamma(C) + 3,$$

with $\text{gon}(C) = \gamma(C) + 2$ if and only if $\gamma(C)$ is computed by a $g^1_{\text{gon}(C)}$.

Definition 2.61. The *Clifford dimension* of a curve C is the integer

$$\text{Cliffdim}(C) := \min\{r(A) \mid A \text{ computes } \gamma(C)\}.$$

We also note that curves with $\text{gon}(C) = \gamma(C) + 3$ are called *exceptional* and are conjectured to be extremely rare [25]. The existence of such curves on K3 surfaces was proven in [25]. Exceptional curves lying on K3 surfaces have been classified by Knutsen, see [49].

Section 2.5

Stability of sheaves on K3 surfaces

In this section, we recall the notions of slope stability (μ-stability, or Mumford-Takemoto stability) and (Gieseker) stability of coherent sheaves. We then focus on these notions on K3 surfaces. Standard references are [42], as well as [41].

We briefly recall standard results on coherent sheaves on a smooth surface S.

- A coherent sheaf F on S is *torsion free* if for every affine open $U \subset S$, $F(U)$ is a torsion free $\mathcal{O}_S(U)$-module.

- If F is torsion free, then F is locally free away from a finite set of closed points.
- The dual, $F^\vee := \mathcal{H}om(F, \mathcal{O}_S)$ of a coherent sheaf F is locally free.
- The double dual $F^{\vee\vee} := \mathcal{H}om(F^\vee, \mathcal{O}_S)$ is called the *reflexive hull* of F. There is a natural morphism $F \to F^{\vee\vee}$ which is injective if and only if F is torsion free. The cokernel of this map is a sheaf with support in dimension zero.
- The rank of a coherent sheaf is the defined to be the rank of $F^\vee$.
- Any torsion free sheaf of rank 1 is isomorphic to a sheaf $M \otimes I_\zeta$ with $M \in \operatorname{Pic}(S)$ and I_ζ the ideal sheaf of a subscheme $\zeta \subset S$ of dimension zero.

Before we can define the notion of stability, we recall some facts about Hilbert polynomials.

For an arbitrary projective scheme X with an ample line bundle $\mathcal{O}_X(1)$, the Hilbert polynomial of a sheaf F is

$$P(F, m) := \chi(F(m)) = \sum_{i=0}^{d} \alpha_i(F) \frac{m^i}{i!},$$

where d is the dimension of the support of F, and $\alpha_i \in \mathbb{Z}$. For a sheaf F of rank r with support of dimension 2 and Chern classes c_1, and c_2 on a smooth projective surface S with ample line bundle H, this becomes

$$\begin{aligned} P(F, m) &= \int_S \operatorname{ch}(F) \left(1 + mH + \frac{m^2 H^2}{2}\right) \operatorname{td}(S) \\ &= \frac{rH^2}{2} m^2 + m\left(H.c_1 + rH.c_1(X)\right) + \alpha_0(F). \end{aligned}$$

Remark 2.62. Note that $\alpha_{\dim(X)}(\mathcal{O}_X)$ is the degree of X with respect to the embedding given by $\mathcal{O}_X(1)$.

Note also that for a sheaf F whose support has maximal dimension $d = \dim(X)$, the rank of F is $\mathrm{rk}(F) = \frac{\alpha_d(F)}{\alpha_d(\mathcal{O}_X)}$. Some authors, e.g., [42], prefer to define the rank of a coherent sheaf in this way.

For sheaves of smaller dimensional support, e.g. torsion sheaves or skyscraper sheaves, the role of torsion-free sheaves is played by *pure sheaves.*

Definition 2.63. A coherent sheaf F of dimension d is called *pure* (of dimension d) if every non-trivial subsheaf $E \subset F$ also has dimension d. In particular, a sheaf of maximal dimension $d = \dim(X)$ is pure if and only if it is torsion free.

Definition 2.64. The *reduced Hilbert polynomial* $p(F, m)$ (sometimes just $p(F)$) of a coherent sheaf F of dimension d is

$$p(F, m) := \frac{P(F, m)}{\alpha_d(F)}.$$

The notion of stability is defined in terms of reduced Hilbert polynomials. In particular, it aims to capture how many sections a sheaf has. While it may seem unmotivated, the notion of stability is crucial to the construction of moduli spaces of sheaves on a given variety. See [41, Chapter 10, Examples 1.1, 1.2] for a discussion of why stability (not just fixing numerical invariants or a Hilbert polynomial) is needed for moduli spaces of sheaves to be well-behaved, in particular to be of finite type and separated.

Recall that polynomials have a natural lexicographic order given by their coefficients. Explicitly, given two polynomials f and g, we say $f \leq g$ if $f(m) \leq g(m)$ for $m \gg 0$. Similarly, we say $f < g$ if $f(m) < g(m)$ for $m \gg 0$.

Definition 2.65. It will be handy to introduce a bit of short-hand notation. If we write *(semi)something* followed by $f(\leq)g$, we mean that $f \leq g$ for semisomething's, whereas $f < g$ for something's.

Definition 2.66 ((Giesker) stability)**.** Let F be a pure sheaf. F is called *(semi)stable* if for all proper non-trivial subsheaves $E \subset F$, one has

$$p(E)(\leq)p(F).$$

Definition 2.67. We say that a sheaf F is *properly semistable* if F is semistable but not stable.

Remark 2.68. If we wish to avoid the need for F to be a pure sheaf, we could define a coherent sheaf of dimension d to be (semi)stable if for all proper subsheaves $E \subset F$, one has

$$\alpha_d(F)P(E)(\leq)\alpha_d(E)P(F).$$

This is however equivalent, see [42, Section 1.2]

Example 2.69. Line bundles are stable. However, a direct sum of two line bundles of different degree is not even semistable.

We recall some standard facts about (semi)stable sheaves.

Proposition 2.70. *Let F be a coherent sheaf of pure dimension d. Then the following are equivalent.*

(i) F is (semi)stable.

(ii) For all proper saturated subsheaves $E \subset F$, one has $p(E)(\leq)p(F)$.

(iii) For all proper quotient sheaves $E \twoheadrightarrow G$ with $\alpha_d(G) > 0$, one has $p(E)(\leq)p(G)$.

(iv) For all proper quotients of pure dimension d, $F \twoheadrightarrow G$, one has $p(F)(\leq)p(G)$.

Proposition 2.71. *Let F and G be semistable coherent sheaves of pure dimension d.*

(i) If $p(F) > p(G)$*, then* $\mathrm{Hom}(F, G) = 0$.

(ii) If $p(F) = p(G)$*, and* $f \in \mathrm{Hom}(F, G)$ *is nontrivial, then* f *is injective if* F *is stable and surjective if* G *is stable.*

(iii) If G *or* F *is stable,* $p(F) = p(G)$*, and* $\alpha_d(F) = \alpha_d(G)$*, then any nontrivial* $f \in \mathrm{Hom}(F, G)$ *is an isomorphism.*

Corollary 2.72. *If* F *is a stable sheaf, then* $\mathrm{End}(F) = \mathrm{Hom}(F, F)$ *is a finite dimensional division algebra over* k*. In particular, if* k *is algebraically closed, then* $k \cong \mathrm{End}(F)$*.*

Definition 2.73. We say a sheaf F is *simple* if $\mathrm{End}(F)$ is a division algebra.

Thus a stable sheaf is simple. The converse is not true. To see a good example, see [42, Example 1.2.10].

For a surface S, the notions of the reduced Hilbert polynomial can be simplified a little. Namely, if F is a pure sheaf of dimension 2 on a smooth projective surface S, then

$$p(F, m) = \frac{m^2}{2} + m\left(\frac{H.c_1(F)}{\mathrm{rk}(F)H^2} + \frac{H.c_1(S)}{H^2}\right) + \frac{\alpha_0(F)}{\mathrm{rk}(F)H^2}.$$

In particular, since the degree 2 term is always the same, and the remaining terms all have H^2 in the denominator, we can normalize the reduced Hilbert polynomial.

Remark 2.74. For a K3 surface, $c_1(S) = 0$, and so for a coherent sheaf F on a polarized K3 surface (S, H), we have

$$p(F, m) = \frac{m^2}{2} + m\frac{H.c_1(F)}{\mathrm{rk}(F)H^2} + \frac{\alpha_0(F)}{\mathrm{rk}(F)H^2}.$$

Definition 2.75. Let F be a coherent sheaf on a polarized K3 surface (S, H). We define the *normalized Hilbert polynomial* of F to be

$$p(F,n) := \frac{\chi(F \otimes H^n)}{\operatorname{rk}(F)} = \frac{H^2}{2!}n^2 + \mu(F)n + \frac{\chi(F)}{\operatorname{rk}(F)}$$

where $\mu(F) = H.c_1(F)/\operatorname{rk}(F)$.

Remark 2.76. Going forward, when we consider sheaves on surfaces, we may take the reduced or the normalized Hilbert polynomials and obtain the same notion of stability. Thus we generally take the normalized Hilbert polynomial for surfaces. However, so that we match the conventions taken in [41, 42], we will take the reduced Hilbert polynomial when we consider arbitrary varieties unless otherwise specified.

Historically, the notion of stability of sheaves first appeared in the study of vector bundles on curves [65]. For a smooth projective curve C of genus g and a locally free sheaf E of rank r on C, the Riemann–Roch theorem states that

$$\chi(E) = \deg(E) + r(1-g).$$

Thus the Hilbert polynomial is

$$P(E,m) = r\deg(C)m + \deg(E) + r(1-g) = r\left(\deg(C)m + \mu(E) + (1-g)\right),$$

where $\mu(E) := \deg(E)/r$ is called the *slope* of E. A notion of stability, see below, can be defined by using the slope instead of Hilbert polynomials, it is called *slope stability*. For further reading on vector bundles on curves, we recommend the classic paper by Atiyah which classifies vector bundles on elliptic curves [5] and the excellent book [66].

The notion of slope stability remains important for varieties of dimension ≥ 2, however, the notions of stability and slope stability differ. For curves, however, the notions of stability coincide, and it may have been more natural to define slope stability first.

Definition 2.77. Let X be a smooth projective variety of dimension d with ample divisor H and F be a coherent sheaf of dimension d. The *degree* of F is

$$\deg(F) := c_1(F).H^{d-1}.$$

The *slope* of F is defined as

$$\mu_H(F) := \frac{\deg(F)}{\mathrm{rk}(F)}.$$

When the choice of H is clear or unnecessary, we omit it from the notation.

Note that the degree, and hence the slope depends on the choice of an ample divisor H. In general, there are many notions of stability, and we direct the interested reader to read [9, 59, 60].

Definition 2.78. A torsion-free sheaf F is called μ-(semi)stable or slope (semi)stable if for all subsheaves $E \subset F$ with $0 < \mathrm{rk}(E) < \mathrm{rk}(F)$ one has

$$\mu(E)(\leq)\mu(F).$$

Example 2.79. As with stability, line bundles are slope stable. The direct sum $F_1 \oplus F_2$ of two slope stable sheaves F_i is never slop stable, but is slope semistable if and only if $\mu(F_1) = \mu_(F_2)$.

Remark 2.80. As with stability, one obtains equivalent conditions for slope stability or semistability as in Proposition 2.71 and Corollary 2.72.

A question jumps out, what is the relationship between stability and slope stability, which the following theorem answers.

Theorem 2.81. *Let X be a variety and E be a pure coherent sheaf on X with support of maximal dimension, then one has the following implications*

$$\mu\text{-stable} \implies \text{stable} \implies \text{semistable} \implies \mu\text{-semistable}.$$

As one would hope, simple objects are the building blocks on larger objects. For sheaves, semistable sheaves are the building blocks for pure sheaves.

Definition 2.82. Let F be a non-trivial pure sheaf of maximum dimension. A *Harder–Narasimhan* filtration for F is an increasing filtration

$$0 = HN_0(F) \subset HN_1(F) \subset \cdots \subset HN_\ell(F) = F$$

such that the factors $gr_i^{HN}(F) = HN_i(E)/HN_{i-1}(F)$ for $i = 1, \ldots, \ell$ are torsion free semistable sheaves with reduced Hilbert polynomials $p_i = p(gr_i^{HN}(F), n)$ satisfying

$$p_{\max}(F) = p_1 > \cdots > p_\ell = p_{\min}(F).$$

In particular, for surfaces, we see that $\mu(gr_1^{HN}(F)) > \mu(gr_2^{HN}(F)) > \cdots > \mu(gr_\ell^{HN}(F))$. Additionally, on surfaces, if F is a vector bundle then the sheaves $HN_i(F)$ are locally free. We also have $\mu(HN_1(F)) > \mu(HN_2(F)) > \cdots > \mu(F)$.

Theorem 2.83. *Every pure sheaf has a unique Harder–Narasimhan filtration.*

Theorem 2.84. *Let F be a pure sheaf. There is a subsheaf $E \subset F$ such that for all subsheaves $G \subset F$, one has $p(E) \geq p(G)$, and equality holds when $G \subset E$. Moreover, the sheaf E is uniquely determined and semistable.*

Definition 2.85. The subsheaf E in the theorem is called the *maximal destabilizing subsheaf of F*.

The uniqueness of the Harder–Narasimhan filtration implies that the polynomials $p_{\max}(F)$ and $p_{\min}(F)$ are well defined.

Example 2.86. F is semistable if and only if F is pure and $p_{\max}(F) = p_{\min}(F)$.

The minimum and maximum reduced Hilbert polynomials play a similar role to the Hilbert polynomials.

Proposition 2.87. *If F and G are pure sheaves with $p_{min}(F) > p_{max}(G)$, then* $\mathrm{Hom}(F, G) = 0$.

Just as semistable objects are building blocks for pure sheaves, we may break down the semistable sheaves into stable pieces, though this may not be unique. This is similar to the Jordan–Hölder decomposition of a group, hence the name.

Definition 2.88. Let F be a semistable sheaf. A *Jordan–Hölder filtration* of F is an increasing filtration

$$0 = JH_0(F) \subset JH_1(F) \subset \cdots \subset JH_\ell(F) = F,$$

such that the factors $gr_i^{JH}(F) = JH_i(F)/JH_{i-1}(F)$ for $i = 1, \ldots, \ell$ are torsion free stable sheaves with reduced Hilbert polynomial $p(F, m)$.

Note that the sheaves $JH_i(F)$ are also semistable with reduced Hilbert polynomial $p(F, m)$. In particular, $\mu(F) = \mu(gr_i^{JH}(F))$ for all i.

The Jordan–Hölder filtration is not uniquely determined, as can be seen by taking the direct sum of two line bundles with the same degree.

Theorem 2.89. *Jordan–Hölder filtrations always exist. Moreover, the associated graded object $gr^{JH}(F) = \bigoplus_i gr_i^{JH}(F)$ is uniquely determined by F.*

Definition 2.90. We say that two semistable sheaves are S-equivalent if the associated graded objects of their Jordan–Hölder filtrations are isomorphic.

We now take a brief detour through some definitions for coherent sheaves on K3 surfaces before stating results on the moduli spaces of sheaves on K3 surfaces.

Recall from the Hirzebruch–Riemann–Roch theorem, Theorem 2.6, that

$$\chi(F) = \int_S \operatorname{ch}(F)\operatorname{td}(S).$$

We can generalize this to an expression for a quadratic form, the Euler pairing.

Definition 2.91. For E and F two coherent sheaves, let

$$\chi(E,F) := \sum_{i\geq 0}(-1)^i \dim \operatorname{Ext}^i(E,F).$$

By Serre duality, we have $\chi(E,F) = \chi(F,E)$, and if E is locally free, then $\chi(E,F) = \chi(E^\vee \otimes F)$. In particular, one can see that $\chi(\mathcal{O}_S, F) = \chi(F)$. We would like to view $\chi(E,F)$ as an intersection, so we want the expression to be somewhat symmetric. The only non-symmetric part is $\operatorname{td}(S)$, so we simply split it into two pieces, $\sqrt{\operatorname{td}(S)} = 1 + \frac{c_2(S)}{24}$. Then Theorem 2.6 generalizes to

$$\chi(E,F) = \int_S \operatorname{ch}^*(E)\operatorname{ch}(F)\operatorname{td}(S) = \int_S \left(\operatorname{ch}^*(E)\sqrt{\operatorname{td}(S)}\right)\left(\operatorname{ch}(F)\sqrt{\operatorname{td}(S)}\right),$$

where ch^* is defined by

$$\operatorname{ch}_i^* := (-1)^i \operatorname{ch}_i$$

which yields

$$\operatorname{ch}^*(E) = \operatorname{ch}(E^\vee)$$

for a locally free sheaf E.

Definition 2.92. For a sheaf E on a polarized K3 surface (S,H), the *Mukai vector* is given by

$$\begin{aligned} v(E) &:= \mathrm{ch}(E)\sqrt{\mathrm{td}(S)} \\ &= (\mathrm{rk}(E),\ c_1(E),\ \mathrm{ch}_2(E) + \mathrm{rk}(E)) \\ &= (\mathrm{rk}(E),\ c_1(E),\ \chi(E) - \mathrm{rk}(E)), \end{aligned}$$

considered as an element in $H^*(S, \mathbb{Z}) = H^0(S, \mathbb{Z}) \oplus H^2(S, \mathbb{Z}) \oplus H^4(S, \mathbb{Z})$.

Remark 2.93. The Mukai vector can also be considered in étale, singular, crystalline, or de Rham cohomology; in the Chow ring; or in the numerical Grothendieck group.

Definition 2.94. The *Mukai pairing* is given by

$$\langle v(E), v(F)\rangle := -\chi(E, F) = -\sum_i (-1)^i \operatorname{Ext}^i(E, F) = -\int_S v(E)^* \wedge v(F),$$

where for $v(E) = v^0 + v^2 + v^4 \in H^*(S, \mathbb{Z})$ with $v^i \in H^i(S, \mathbb{Z})$, we write $v(E)^* := v^0 - v^2 + v^4$.

Remark 2.95. Note that the Mukai pairing differs from the usual intersection form on $H^*(S, \mathbb{Z})$ by a sign on $H^0 \oplus H^4$.

Proposition 2.96. *For a simple sheaf F, we have $\chi(F, F) = 2 - \dim \operatorname{Ext}^1(F, F) \leq 2$. In particular, $\langle v(F), v(F)\rangle \geq -2$.*

Proof. Indeed, since F is simple, we have $\operatorname{Ext}^0(F, F) = \operatorname{Hom}(F, F) \cong k$, and

$$\operatorname{Ext}^2(F, F) \cong \operatorname{Hom}(F, F)^\vee \cong k.$$

□

Theorem 2.97. *For fixed Hilbert polynomial P, the functor taking a k-scheme X of finite type to the set of families of semistable flat sheaves on S with Hilbert polynomial*

P parameterized by X,

$$\begin{aligned}\mathcal{M}:\ & (Sch/k)^{fin} \to (Sets)\\ & X \mapsto \{E \in \mathrm{Coh}(X \times S) \ \mid \ E \text{ is } X\text{-flat}, P(E_x) = P, E_x \text{ is semistable}\}/\sim,\end{aligned}$$

is corepresented by a projective k-scheme M, where $E \sim E \otimes \pi_1^* L$ for $\pi_1 : X \times S \to X$ is projection onto X and $L \in \mathrm{Pic}(X)$ is any line bundle. That is, $\mathcal{M}$ has a moduli space M. Moreover, the closed points of M parameterize the S-equivalence classes of semistable sheaves with Hilbert polynomial P.

Proposition 2.98. *Let M be the moduli space of $\mathcal{M}$ and let $t \in M$ be a point corresponding to a stable sheaf $F \in \mathcal{M}(k)$. Then there is a natural isomorphism $T_t M \cong \mathrm{Ext}^1(F, F)$.*

Since $P(F, m) = \chi(F(m)) = -\langle v(F), v(\mathcal{O}_S(-m))\rangle$, the Mukai vector determines the Hilbert polynomial.

Conversely, if E is a sheaf on $X \times S$ with X connected, such that E_x is flat over S_x for every $x \in X$; that is, E is a X-flat family of sheaves E_x on S parameterized by X, then $v(E_x)$ is constant. Indeed, $\chi(E_x, F)$ is constant for all coherent sheaves F on S.

Thus, when we consider moduli spaces of sheaves on K3 surfaces, instead of fixing the Hilbert polynomial, it is more convenient to fix a Mukai vector. So let v be a Mukai vector, and consider the moduli functor $\mathcal{M}(v)$ is semistable sheaves with Mukai vector v, and its moduli space $M(v)$. Since the Mukai vector depends on the polarization H, we write $M_H(v)$ if we wish to specify the polarization. There is a (possibly empty) subscheme $M^s(v) \subset M(v)$ parameterizing stable sheaves with Mukai vector v.

Theorem 2.99. *At a point $t \in M^s(v)$ corresponding to a sheaf F, the moduli space*

$M^s(v)$ is smooth of dimension

$$\dim M^s(v) = \dim \operatorname{Ext}^1(F, F).$$

Corollary 2.100. *Either $M^s(v)$ is empty, or it is a smooth quasi-projective variety of dimension $2 + \langle v, v\rangle$.*

For more details and examples, we suggest [42] and [41, Chapters 9-10].

Chapter 3

Brill–Noether theory with K3 surfaces

In this chapter, we give historical background of the main results in Brill–Noether theory on K3 surfaces, and highlight the major contributions of K3 surfaces to the study of Brill–Noether theory of curves. In Section 3.2, we recall properties of Lazarsfeld–Mukai bundles and outline Lazarsfeld's proof of the Brill–Noether–Petri theorem. In Section 3.3, we review Lelli-Chiesa's generalized Lazarsfeld–Mukai bundles.

Section 3.1 Background

Many results in Brill–Noether theory are proved using degeneration techniques, see Section 2.4. For instance, Gieseker's proof of the Brill–Noether–Petri theorem used a careful consideration of degenerations to stable curves, as well as elaborate combinatorial arguments. Techniques developed by Lazarsfeld and Green–Lazarsfeld introduced Lazarsfeld–Mukai bundles on K3 surfaces as incredibly useful objects to study linear systems on curves on K3 surfaces. Before defining and stating properties of Lazarsfeld–Mukai bundles, we recall some theorems which were proved using these techniques. We conclude this chapter by recalling the notion of generalized Lazarsfeld–Mukai bundles introduced by Lelli-Chiesa.

We restate Theorem 2.52 for convenience.

Theorem 3.1 (Brill–Noether–Petri Theorem, [31])**.** *For a general curve C, all line bundles on C have injective Petri map μ_0.*

Lazarsfeld proved a slightly different theorem, with an added assumption on the linear system of the curve on a K3 surface.

Theorem 3.2 ([53])**.** *Let S be a complex projective K3 surface, and let $C_0 \subset S$ be a smooth connected curve. If every divisor in the linear system $|C_0|$ is reduced and irreducible, then the general curve $C \in |C_0|$ satisfies Petri's condition.*

The hypothesis is satisfied, in particular, if $\operatorname{Pic}(S)$ has rank 1 and is generated by $[C_0]$ (written $\operatorname{Pic}(S) = \mathbb{Z}[C_0]$) and in some cases when S has Picard rank 2, see [2]. Since for any genus $g \geq 2$, there exists a K3 surface of genus g with $\operatorname{Pic}(S) = \mathbb{Z}[C_0]$ for some $C_0 \subset S$, Lazarsfeld's theorem implies the Brill–Noether–Petri theorem since by semicontinuity, only a single curve in each genus suffices to prove the theorem in general. In the course of Lazarsfeld's proof, it is also shown that Brill–Noether general curves are readily found on K3 surfaces, providing a new proof of a special case of the Brill–Noether theorem.

Theorem 3.3 ([53, Corollary 1.4])**.** *If every member of the linear series $|C_0|$ is reduced and irreducible, then $\rho(C, A) \geq 0$ for every smooth curve $C \in |C_0|$ and every line bundle A on C.*

The study of special divisors on curves on K3 surfaces was considered by Saint-Donat, Reid, and others [21, 32, 44, 46, 56, 57, 61, 75, 77]. Following classical work on linear systems of type g^1_d, Harris and Mumford conjectured that the gonality of curves on K3 surfaces should remain constant in a linear system. After a counterexample was constructed by Donagi and Morrison, the conjecture was modified by Green, to a statement that the Clifford index of curves should remain constant in a linear system on a K3 surface. We discuss this in more detail in Chapter 4.

Recall that the Clifford index of a curve is the integer

$$\gamma(C) = \min\{\gamma(g^r_d) = d - 2r \mid \ C \text{ admits a } g^r_d \text{ with } r \geq 1, g - d + r \geq 2\},$$

which is roughly a measure of the gonality-2, see Section 2.4 for more details.

That the Clifford index is constant in linear systems on K3 surfaces was proven by Green and Lazarsfeld using Lazarsfeld–Mukai bundles on K3 surfaces.

Theorem 3.4 (Green-Lazarsfeld [32])**.** *Let S be a K3 surface and $C \subset S$ a smooth irreducible curve of genus $g \geq 2$. Then $\gamma(C') = \gamma(C)$ for any smooth curve $C' \in |C|$. Moreover, if $\gamma(C) < \lfloor \frac{g-1}{2} \rfloor$, then there is a line bundle $L \in \operatorname{Pic}(S)$ such that $L|_{C'}$ computes $\gamma(C')$.*

The second conclusion concerning the restriction of line bundles on K3 surfaces to line bundles on curves has been an area of interest as well. Saint-Donat [77] and Reid [75] investigated when a linear system of type g^1_d on a curve $C \in |H|$ on a polarized K3 surface (S, H) is the restriction of a line bundle $L \in \operatorname{Pic}(S)$ to C. Donagi and Morrison proved a more general result for g^1_d's [21], and made a conjecture regarding the general situation, see Section 4.2. Lelli-Chiesa has provided a proof in the case $r = 2$ [56], and when A computes the Clifford index of C [57] improving the theorem by Green–Lazarsfeld. More recently, we have provided results concerning the case $r = 3$ [6], see Theorem 5.15. We explore these results more in Chapter 4.

Section 3.2

Lazarsfeld–Mukai bundles

In this section, we introduce Lazarsfeld–Mukai bundles and establish some of their basic properties. We also outline Lazarsfeld's proof of Theorem 3.3. We follow [6, 56, 73], other references include [53, 32, 41].

Let S be a K3 surface and $\iota\colon C \hookrightarrow S$ be a smooth irreducible curve of genus g. Any basepoint free linear series $A \in W^r_d(C) \setminus W^{r+1}_d(C)$ can be considered as a globally generated sheaf $\iota_*(A)$ on S. Therefore, the evaluation map

$$H^0(C, A) \otimes \mathcal{O}_S \to \iota_*(A)$$

is surjective, and the kernel is a Lazarsfeld–Mukai bundle. Some authors refer to the kernel of the evaluation map as a *kernel bundle*.

Definition 3.5. The *Lazarsfeld–Mukai bundle* $F_{C,A}$ on S associated to C and A is defined by

$$F_{C,A} := \ker\left(H^0(C, A) \otimes \mathcal{O}_S \to \iota_*(A)\right).$$

$$0 \to F_{C,A} \to H^0(C, A) \otimes \mathcal{O}_S \to \iota_*(A) \to 0.$$

Its dual, $E_{C,A} := F^\vee_{C,A}$ is also called a *Lazarsfeld–Mukai (LM) bundle*.

We focus mainly on the LM bundle $E_{C,A}$, and we say that $E_{C,A}$ is the LM bundle associated with the pair (C, A), or the LM bundle associated to A on C. Before we establish basic properties of $E_{C,A}$, we need a quick lemma.

Lemma 3.6 ([73, Lemma 2.5])**.** *Let D be a Cartier divisor on a smooth variety X. Then*

$$\mathcal{E}xt^1(\mathcal{O}_D, \mathcal{O}_X) \cong \mathcal{O}_D(D).$$

Proof. Since D is a divisor, $\mathcal{O}_D$ is a torsion sheaf on X, thus $\mathcal{O}_D^\vee = 0$. Consider the exact sequence for the divisor D,

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0,$$

and dualize (apply $\mathcal{H}om(\cdot, \mathcal{O}_X)$) to get

$$0 \to \underbrace{\mathcal{H}om(\mathcal{O}_D, \mathcal{O}_X)}_{0} \to \mathcal{O}_X \to \underbrace{\mathcal{H}om(\mathcal{O}_X(-D), \mathcal{O}_X)}_{\mathcal{O}_X(D)} \to \mathcal{E}xt^1(\mathcal{O}_D, \mathcal{O}_X) \to \underbrace{\mathcal{E}xt^1(\mathcal{O}_X, \mathcal{O}_X)}_{0}.$$

Tensoring the short exact sequence for D with $\mathcal{O}_X(D)$, we obtain

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \mathcal{O}_D(D) \to 0.$$

Comparing the last two short exact sequences thus shows $\mathcal{E}xt^1(\mathcal{O}_D, \mathcal{O}_X) \cong \mathcal{O}_D(D)$. □

Lemma 3.7 ([73, Lemma 2.6])**.** *The LM bundle $F_{C,A}$ is a vector bundle on S of rank $r + 1 = h^0(C, A)$. The LM bundle $E_{C,A}$ sits in a short exact sequence*

$$0 \to H^0(C, A)^\vee \otimes \mathcal{O}_S \to E_{C,A} \to \iota_*(\omega_C \otimes A^\vee) \to 0.$$

Proof. Since A is a torsion sheaf on S, we immediately see $\mathrm{rk}(F_{C,A}) = r + 1$ from the short exact sequence defining $F_{C,A}$.

Since $(\iota_* A)^\vee = 0$ and $\mathcal{E}xt^1(H^0(C, A) \otimes_{\mathcal{O}_S}, \mathcal{O}_S) = 0$, dualizing the short exact sequence defining $F_{C,A}$, we obtain

$$0 \to H^0(C, A)^\vee \otimes \mathcal{O}_S \to E_{C,A} \to \mathcal{E}xt^1(\iota_* A, \mathcal{O}_S) \to 0,$$

as in the previous lemma. Finally, we see that

$$\mathcal{E}xt^1(\iota_*(A), \mathcal{O}_S) \cong \mathcal{E}xt^1(\mathcal{O}_C, \mathcal{O}_S) \otimes \iota_*(A^\vee)$$

by a similar argument as in [38, Chapter III, Proposition 6.7], and $\mathcal{E}xt^1(\mathcal{O}_C, \mathcal{O}_S) \cong \iota_*(\mathcal{O}_C(C)) \cong \iota_*(\omega_C)$ by the lemma above and adjunction. Thus $\mathcal{E}xt^1(A, \mathcal{O}_S) \cong \iota_*(\omega_C \otimes A^\vee)$, as desired. □

The most important properties of $F_{C,A}$ and $E_{C,A}$ can now be read off from the short exact sequences.

Proposition 3.8. *Let C be a smooth irreducible curve of genus g and let $A \in \mathrm{Pic}(C)$ be of type g^r_d. Let $E_{C,A}$ be the LM bundle associated to (C, A). Then*

(i) $\mathrm{rk}(E_{C,A}) = r + 1$*;*

(ii) $\det(E_{C,A}) = \mathcal{O}_S(C)$*, i.e.* $c_1(E_{C,A}) = C$*, and* $c_2(E_{C,A}) = \deg(A)$*;*

(iii) *$E_{C,A}$ is globally generated off the base locus of $\omega_C \otimes A^\vee$ (a finite set);*

(iv) $H^0(S, E_{C,A}) = h^0(C, A) + h^0(C, \omega_C \otimes A^\vee) = r + 1 + g - d + r$*;*

(v) $h^1(S, E_{C,A}) = h^1(S, F_{C,A}) = 0$*;*

(vi) $h^2(S, E_{C,A}) = h^0(S, F_{C,A}) = 0$*;*

(vii) $\chi(S, \mathcal{E}nd(E_{C,A})) = 2 - 2\rho(g, r, d) = 2h^0(S, E_{C,A} \otimes F_{C,A}) - h^1(S, E_{C,A} \otimes F_{C,A})$*; and*

(viii) *if $\rho(g, r, d) < 0$, then $h^0(S, \mathcal{E}nd(E_{C,A})) > 1$ ($E_{C,A}$ is not simple), hence $E_{C,A}$ is not stable.*

Proof. To prove (i) we use the fact that $F_{C,A}$ has rank $r + 1$.

The remaining properties of $E_{C,A}$ follow by applying the Whitney formula to the short exact sequence

$$0 \to H^0(C,A)^\vee \otimes \mathcal{O}_S \to E_{C,A} \to \iota_*(\omega_C \otimes A^\vee) \to 0,$$

or by taking cohomology and using the fact that $H^1(S,\mathcal{O}_S) = 0$.

The Chern classes of $\iota_*(\omega_C \otimes A^\vee)$ are readily computed by following [30, Lemma 1, page 30]. We have $c_1(\iota_*(\omega_C \otimes A^\vee)) = C$, and

$$c_2(\iota_*(\omega_C \otimes A^\vee)) = C^2 - \deg(\omega_C \otimes A^\vee) = \deg(A).$$

Applying Whitney's formula, we see that

$$1 + c_1(E_{C,A}) + c_2(E_{C,A}) = 1 \cdot (1 + c_1(\iota_*(\omega_C \otimes A^\vee)) + c_2(\iota_*(\omega_C \otimes A^\vee))),$$

and the rest follows.

To prove (iii), we take cohomology and see that we have a surjection

$$H^0(S,E_{C,A}) \twoheadrightarrow H^0(S,\iota_*(\omega_C \otimes A^\vee)).$$

Thus every section of $\iota_*(\omega_C \otimes A^\vee)$ lifts to a section of $E_{C,A}$, proving (iii).

Similarly, (iv) and (v) are proven by taking cohomology and noting

$$h^1(S,\iota_*(\omega_C \otimes A^\vee)) = h^2(S,\iota_*(\omega_C \otimes A^\vee)) = h^1(S,\mathcal{O}_S) = 0,$$

and applying Serre duality to see $h^1(S,E_{C,A}) = h^1(S,F_{C,A})$.

To prove (vi), we first note that $H^0(S,F_{C,A}) = 0$ which can be seen by taking cohomology in the short exact sequence defining $F_{C,A}$ and recalling that

$$H^0(S, \iota_*(A)) = H^0(C, A).$$

Applying Serre duality to $F_{C,A}$ shows that

$$0 = H^0(S, F_{C,A}) = H^2(S, E_{C,A})^\vee.$$

To prove (*vii*), we compute using the Mukai pairing, to obtain

$$\begin{aligned}\chi(S, \mathcal{E}nd(E_{C,A})) &= \chi(S, E_{C,A} \otimes F_{C,A}) \\ &= rc_1(E_{C,A})^2 - 2(r+1)c_2(E_{C,A}) + 2(r+1)^2 \\ &= 2 - 2\rho(g, r, d),\end{aligned}$$

and since $\mathcal{E}nd(E_{C,A})$ is self-dual Serre duality gives $h^0(S, \mathcal{E}nd(E_{C,A})) = h^2(S, \mathcal{E}nd(E_{C,A}))$, from which the last equality follows.

Finally, to see (*viii*), we see from (*vii*) that if $\rho(g, r, d) < 0$, then $\chi(S, \mathcal{E}nd(E_{C,A})) > 2$, hence $h^0(S, \mathcal{E}nd(E_{C,A})) = \dim \mathrm{End}(E_{C,A}) > 1$, which over an algebraically closed field shows that $E_{C,A}$ is not simple. □

We are now ready to outline a proof of Theorem 3.3. We follow a few statements in [41, Chapter 9, Section 2]. For the remainder of the section, let C be a smooth curve on a K3 surface S, and A a basepoint free (globally generated) line bundle on C unless otherwise specified.

Proposition 3.9 ([41, Section 9.2, Proposition 2.2]). *Assume that $\omega_C \otimes A^\vee$ is globally generated and that every curve in the linear system $|C|$ is reduced and irreducible. Then $F_{C,A}$ is locally free and simple.*

Proof. Clearly $F_{C,A}$ is simple if and only if its dual, $E_{C,A}$, is simple. Since $\omega_C \otimes A^\vee$ is globally generated, it is basepoint free, hence $E_{C,A}$ is globally generated.

If $E_{C,A}$ is not simple, then there exists a non-trivial endomorphism $\varphi \in \operatorname{End}(E_{C,A})$ with non-trivial kernel. Let $K = \operatorname{im}\varphi$, which sits in a short exact sequence

$$0 \to K \to E_{C,A} \to E_{C,A}/K \to 0,$$

hence K is torsion free of rank $0 < \operatorname{rk}(K) < r+1$ since φ is not trivial.

Since $E_{C,A}$ is globally generated, and K and $E_{C,A}/K$ are both quotients of $E_{C,A}$, their determinants are globally generated and hence $\det(K) \cong \mathcal{O}_S(C_1)$ and $\det(E_{C,A}/K) \cong \mathcal{O}_S(C_2)$ for some effective curves $C_1, C_2 \subset S$. We claim that $\mathcal{O}_S(C_1)$ and $\mathcal{O}_S(C_2)$ are both nontrivial.

We first prove that $\mathcal{O}_S(C_1)$ is not trivial. Indeed, it is easy to see that $\operatorname{Hom}(K, \mathcal{O}_S) = 0$ and K is globally generated since K is a quotient of $E_{C,A}$. The restriction of K to a generic ample curve D is locally free and globally generated. Thus, there is an exact sequence

$$0 \to (K|_D)^* \to \mathcal{O}_D^{\operatorname{rk}(K)+1} \to \det(K)|_D \to 0$$

of vector bundles on D, see [42, Chapter 5]. For sufficiently positive D, the restriction map $\operatorname{Hom}(K, \mathcal{O}_S) \to \operatorname{Hom}(K|_D, \mathcal{O}_D)$ is surjective whereby $\operatorname{Hom}(K|_D, \mathcal{O}_D) = 0$ and thus

$$H^0(D, (K|_D)^*) = H^0(K|_D, \mathcal{O}_D) = 0.$$

Hence $h^0(D, \det(K)|_D) \geq \operatorname{rk}(K)+1$, thus $\deg(K|_D) > 0$, and thus $\deg(\det(K)|_D) > 0$, whereby $\det(K) = \mathcal{O}_S(C_1)$ is not trivial.

To prove that $\mathcal{O}_S(C_2)$ is not trivial, apply the same argument to $(E_{C,A}/K)/\operatorname{Tors}(E_{C,A}/K)$, since $E_{C,A}/K$ is not necessarily torsion free.

Now, since $\det(E_{C,A}) \cong \mathcal{O}_S(C)$, we have

$$\mathcal{O}_S(C_1 + C_2) \cong \det(K) \otimes \det(E_{C,A}/K) \cong \mathcal{O}_S(C),$$

thus $C_1 + C_2 \in |C|$, which contradicts the assumption that every curve in $|C|$ is reduced and irreducible. Thus $F_{C,A}$ and $E_{C,A}$ must be simple. □

Remark 3.10. It turns out that in this case $E_{C,A}$ is in fact μ-stable, which we prove in Corollary 3.13.

Corollary 3.11 ([41, section 9.2, Corollary 2.4])**.** *Assume that every curve in $|C|$ is reduced and irreducible, then every line bundle $A \in \mathrm{Pic}(S)$ satisfies $\rho(C, A) \geq 0$.*

Proof. First assume that A and $\omega_C \otimes A^\vee$ are both globally generated. From part (*vii*) of Proposition 3.8, we see that if $F_{C,A}$ is simple, then $\rho(C, A) \geq 0$.

Thus it remains to reduce to the case that A and $\omega_C \otimes A^\vee$ are globally generated. If $h^0(C, A) = 0$ or $h^1(C, A) = 0$, then

$$\rho(C, A) = g - h^0(C, A)h^1(C, A) = g \geq 0,$$

and the result follows. Suppose that A is not globally generated, but $h^0(C, A) \neq 0$, and let D be the fixed locus of A. Hence $A(-D)$ is globally generated (basepoint free), $h^0(C, A) = h^0(C, A(-D))$, and

$$h^1(C, A) = h^0(C, \omega_C \otimes A^\vee) \leq h^0(C, \omega_C \otimes A^\vee(D)) = h^1(C, A(-D)).$$

Therefore, $\rho(C, A) \geq \rho(C, A(-D))$. Thus we may assume that A is globally generated. Likewise, we may assume that $\omega_C \otimes A^\vee$ is globally generated and in removing the basepoints of $\omega_C \otimes A^\vee$ we do not introduce basepoints for A, otherwise ω_C would not be globally generated. □

Corollary 3.12 ([41, section 9.2, Corollary 2.5])**.** *Assume that every curve in $|C|$ is reduced and irreducible. If $\rho(g, r, d) < 0$, then $W^r_d(C) = \varnothing$.*

Proof. We prove the contrapositive. Suppose $A \in W^r_d(C)$. Then $h^0(C, A) \geq r + 1$ and $\deg(A) = d$. Thus by Theorem 2.44, we have $h^1(C, A) \geq g - d + r$, and thus $\rho(g, r, d) = \rho(C, A) \geq 0$ by the previous corollary. □

Corollary 3.13 ([41, Section 9.3, Corollary 3.3])**.** *If $\mathcal{O}_S(C)$ generates* Pic(S) *and $\omega_C \otimes A^\vee$ is basepoint free, then $F_{C,A}$ (and hence $E_{C,A}$) is μ-stable.*

Proof. If $F \subset F_{C,A}$ is a locally free subsheaf of rank s, then

$$\det(F) \subset \bigwedge^s F_{C,A} \subset \bigwedge^s \mathcal{O}_S^{r+1} = \mathcal{O}_S^n.$$

Thus $\mathcal{O}_S \subset \det(F)^\vee$, so $h^0(S, \det(F)^\vee) > 0$. As in the proof of Proposition 3.9 since $E_{C,A}$ is globally generated, we have that $\det(F)^\vee \cong \mathcal{O}_S(C_1)$ with $\mathcal{O}_S(C_1)$ nontrivial.

With the additional assumption that $\mathrm{Pic}(S)$ is generated by $\mathcal{O}_S(C)$, the line bundle $\mathcal{O}_S(C)$ is automatically ample and the slope μ_H is taken with respect to $H = C$. Thus if $F \subset F_{C,A}$ is a locally free sheaf of strictly smaller rank, then $\det(F)^\vee \cong \mathcal{O}_S(kC)$ for some $k > 0$, as $\det(F)^\vee$ has global sections. Hence $\deg(F) = k \deg(F_{C,A}) < 0$, thus

$$\frac{-kC^2}{s} = \mu(F) < \mu(F_{C,A}) = \frac{-C^2}{r+1}$$

as $s < r + 1$. □

Section 3.3

Generalized Lazarsfeld–Mukai bundles

In [57], Lelli-Chiesa introduces generalized LM bundles and shows their utility in studying the Brill–Noether theory of curves on K3 surfaces. In particular, the proof of Lelli-Chiesa's strengthening of Theorem 3.4, [57, Theorem 4.2], relies on studying destabilizing subsheaves of generalized Lazarsfeld–Mukai bundles. We recall the definition and basic properties, and return to Lelli-Chiesa's theorem in the next chapter once we introduce more context.

Definition 3.14. Let C be a curve and $A \in \mathrm{Pic}(C)$. The linear system $|A|$ is called *primitive* if both A and $\omega_C \otimes A^\vee$ are basepoint free.

Definition 3.15 ([57] Definition 1)**.** A torsion free coherent sheaf E on S with $h^2(S, E) = 0$ is called a *generalized Lazarsfeld–Mukai bundle* (gLM bundle) of type (I) or (II), respectively, if

(I) E is locally free and generated by global sections off a finite set;

or

(II) E is globally generated.

Remark 3.16 ([57] Remark 1)*.* If conditions (I) and (II) of Definition 3.15 are both satisfied, then E is the LM bundle associated with a smooth irreducible curve $C \subset S$ and a primitive linear series (A, V) on C, i.e. $E = E_{C,(A,V)}$, where $E_{C,(A,V)}$ is the dual of the kernel of the evaluation map $V \otimes \mathcal{O}_S \to A$. Furthermore, $V = H^0(C, A)$ if and only if $h^1(S, E) = 0$, in which case E is just the LM bundle associated to A.

Definition 3.17. Let E be a gLM bundle. The *Clifford index of E* is:

$$\gamma(E) := c_2(E) - 2(\mathrm{rk}(E) - 1).$$

Remark 3.18*.* For the LM bundle $E_{C,A}$ for a smooth curve $C \subset S$ and A a g^r_d on C, one has $\gamma(E_{C,A}) = \gamma(A)$ by Proposition 3.8.

Remark 3.19*.* If E is a gLM bundle of type (I), then there is a short exact sequence

$$0 \to \widetilde{E} \to E \to \tau \to 0$$

where $\widetilde{E}$ is a globally generated subsheaf of E satisfying $H^0(S, \widetilde{E}) = H^0(S, E)$ and τ is a 0-dimensional sheaf supported on the points at which E is not globally generated.

If E is a gLM bundle of type (II), then there is a short exact sequence

$$0 \to E \to E^{\vee\vee} \to \kappa \to 0,$$

where κ is a 0-dimensional sheaf on S such that $H^0(S, E)$ generates E off the support of κ.

Proposition 3.20 ([57, Proposition 2.4])**.** *Let E be a gLM bundle such that $c_1(E)^2 > 0$. If E is of type (I), then the following inequality is satisfied:*

$$\mathrm{Cliff}(E) \geq 2h^1(E) + l(\tau),$$

where $\ell(\tau)$ is the length of the 0-dimensional sheaf τ appearing in Remark 3.19. If instead E is of type (II), we have:

$$\mathrm{Cliff}(E) \geq \mathrm{Cliff}(E^{\vee\vee}) + l(\kappa),$$

where κ is the 0-dimensional sheaf appearing in Remark 3.19.

Lemma 3.21 ([57] Corollary 2.5)**.** *Let E be a gLM bundle of rank r and $c_1(E)^2 > 0$. Then, $\gamma(E) \geq 0$. Furthermore, $\gamma(E) = 0$ only in the following cases:*

(i) *$r = 1$ and E is a globally generated line bundle;*

(ii) *$E = E_{C,\omega_C}$ for some smooth irreducible curve $C \subset S$ of genus $g = r \geq 2$;*

(iii) *$r > 1$ and $E = E_{C,(r-1)g^1_2}$ for some smooth hyperelliptic curve $C \subset S$ of genus $g > r$.*

Regarding part (*iii*) of Corollary Lemma 3.21, the classification of hyperelliptic linear systems on S due to Saint-Donat ([77, Theorem 5.2]) is helpful.

Theorem 3.22 ([57, Theorem 2.6])**.** *Let $C \subset S$ be a smooth hyperelliptic curve of genus $g \geq 2$ and set $L := \mathcal{O}_S(C)$. Then, one of the following occurs:*

- *The equality $c_1(L)^2 = 2$ holds.*
- *There is a smooth, irreducible curve $B \subset S$ of genus 2 satisfying $L \simeq \mathcal{O}_S(2B)$.*
- *There exists an irreducible elliptic curve $\Sigma \subset S$ such that $c_1(L) \cdot \Sigma = 2$.*

This finishes the basic facts about gLM bundles with $c_1^2 > 0$. The case of gLM bundles with $c_1^2 = 0$ is simpler.

Proposition 3.23 ([57, Proposition 2.7])**.** *Let E be a gLM bundle such that $c_1(E)^2 = 0$. Then, E is both locally free and globally generated and satisfies $c_2(E) = 0$. Furthermore, if $h^1(E) = 0$, then $E = \mathcal{O}_S(\Sigma)^{\oplus \operatorname{rk}(E)}$ for an irreducible elliptic curve $\Sigma \subset S$.*

We conclude this chapter with a few useful facts about gLM bundles.

Lemma 3.24. *Let $N \in \operatorname{Pic}(S)$ be nontrivial and globally generated with $h^0(S, N) \neq 0$. Let $E = E_{C,A}$ and suppose we have a short exact sequence*

$$0 \longrightarrow N \longrightarrow E \longrightarrow E/N \longrightarrow 0$$

with E/N torsion free. Then E/N satisfies $h^1(S, E/N) = h^2(S, E/N) = 0$. If A is primitive, then E/N is a gLM bundle of type (II). If we further assume that E/N is locally free, then it is a LM bundle for a smooth irreducible curve $D \in |H - N|$. If A is not primitive and E/N is assumed locally free, then E/N is a gLM bundle of type (I). In any of the above cases, we have

- *$c_1(E/N) = H - N$;*

- $c_2(E/N) = d + N^2 - H.N$;
- $\gamma(E/N) = \gamma(E_{C,A}) + N^2 - H.N + 2$.

Proof. If A is primitive, we see that E/N is globally generated as E is globally generated. From the long exact sequence in cohomology, and noting that $h^2(S, N) = h^1(S, E) = h^2(S, E) = 0$, we see that $h^1(S, E/N) = h^2(S, E/N) = 0$. Thus E/N is a gLM bundle of type (II). If E/N is assumed to be locally free, then as in Remark 3.16, $E/N = E_{D,B}$ is the LM bundle associated to a smooth irreducible curve $D \subset S$ and a line bundle B on D. Finally, if A is not primitive, then E/N is globally generated off a finite set as it is the quotient of E, which is also globally generated off a finite subset. Thus E/N is a gLM of type (I).

Applying Whitney's formula to the exact sequence, we see that

$$1 + c_1(E) + c_2(E) = (1 + c_1(E/N) + c_2(E/N))\,(1 + N)\,,$$

hence $c_1(E/N) = H - N$ and $c_2(E/N) = d + N^2 - H.N$. Finally, as $\gamma(E/N) = c_2(E/N) - 2(\mathrm{rk}(E/N) - 1)$ and $\mathrm{rk}(E/N) = \mathrm{rk}(E) - 1 = (r+1) - 1 = r$, it follows that

$$\gamma(E/N) = d + N^2 - H.N - 2(r-1) = d - 2r + N^2 - H.N + 2 = \gamma(E) + N^2 - H.N + 2.$$

□

Remark 3.25. If A is of type g^r_d and $L = H - N$ is a lift of A with $L^2 = 2r - 2$, then the last equality gives $\gamma(E/N) = \gamma(A) + (2r-2) - d + 2 = 0$.

Remark 3.26. The same argument as above shows that if A is primitive and $M \subset E = E_{C,A}$ is a subsheaf such that E/M is torsion free (e.g. obtained through a Harder–Narasimhan filtration), then E/M is a gLM bundle of type (II). Moreover, by [57, Proposition 2.7], if $c_1(E/M)^2 = 0$, then $c_2(E/M) = 0$. In the following sections, we will use the contrapositive of this when $c_2(E/M) > 0$.

We give a brief summary of gLM bundles of low Clifford index. Such a characterization can be useful in eliminating certain types of filtrations of Lazarsfeld–Mukai bundles, see Section 6.3.6.

Proposition 3.27. *Let $E = E_{C,A}$ be a LM bundle associated to a primitive linear system A on $C \subset S$. Suppose there is a globally generated saturated line bundle $N \subset E$ with $h^0(S, N) \geq 2$ and $\gamma(E/N) \leq 2$. Then either $c_1(E/N)^2 = 0$ in which case $E/N = \mathcal{O}_S(\Sigma)^{\oplus r(A)}$ for an irreducible elliptic curve $\Sigma \subset S$, or $c_1(E/N)^2 > 0$ and one of the following holds:*

(i) $\gamma(E/N) = 0$*, hence E/N is a LM bundle;*

(ii) $(E/N)^{\vee\vee}$ *is a LM bundle of Clifford index* 0*;*

(iii) E/N *or* $(E/N)^{\vee\vee}$ *is a LM bundle of Clifford index* 1*;*

(iv) E/N *is a LM bundle of Clifford index* 2*.*

Proof. By Lemma 3.24, we see that E/N is a gLM bundle of type (II). If $c_1(E/N)^2 = 0$, [57, Proposition 2.7] gives $E/N = \mathcal{O}_C(\Sigma)^{\oplus r(A)}$, as stated.

We now assume $c_1(E/N)^2 > 0$.

If $\gamma(E/N) = 0$, we are in case (i). The fact that E/N is a LM bundle follows from Proposition 3.20, as then E/N is globally generated as $\tau = \varnothing$.

If $\gamma(E/N) = 1$ and E/N is locally free, we are in case (iii). If E/N is not locally free, then [57, Proposition 2.4] shows that $(E/N)^{\vee\vee}$ has Clifford index 0, and we are in case (ii).

If $\gamma(E/N) = 2$ and E/N is locally free, we are in case (iv). If (E/N) is not locally free, then [57, Proposition 2.4] again shows that $(E/N)^{\vee\vee}$ has Clifford index 0 or 1, and we are in case (ii) or (iii), respectively. □

Remark 3.28. Furthermore, in case (*iv*) above, fixing the rank of A narrows the possibilities for the classification of E/N. For example, when A has rank 3 and E/N has Clifford index 2, then $E/N = E_{D,g^2_6}$ for a g^2_6 on a smooth curve D in the linear system of $\det(E/N)$.

Likewise, restricting the Clifford index of a LM bundle E similarly restricts to which linear system E corresponds. For example, if E is a LM bundle and $\gamma(E) = 1$ or $\gamma(E) = 2$, then a smooth irreducible curve $D \in |\det(E)|$ has $\gamma(D) \leq 2$ and is thus either hyperelliptic (when $\gamma(D) = 0$), trigonal or a plane quintic (when $\gamma(D) = 1$), or a plane sextic (when $\gamma(D) = 2$ and $\mathrm{rk}(E) = 3$).

One could similarly characterize gLM bundles of type (II) of higher Clifford index, using [57, Proposition 2.4] repeatedly as in Proposition 3.27, and then fixing the rank as above.

Chapter 4

Lifting Linear Systems from curves to K3 surfaces

The study of linear systems on curves on K3 surfaces has a rich history. In particular, the linear systems are well behaved as the curve varies in its linear equivalence class on the K3. For instance, Saint-Donat proved in [77] that for a curve $C \subset S$, C admits a g^1_2 or a g^1_3 if and only if every smooth curve $C' \in |C|$ does as well. Following this, Reid [75] found that there are similar results for other line bundles of type g^1_d. However, Donagi and Morrison give conterexamples to the propogation of g^1_d's. The general question of persisting g^r_d's in linear systems arose out of work of Harris and Mumford [37] where they used the existence of certain divisors in $\mathcal{M}_g$ to give bounds on the Kodaira dimension. These techniques have been extended by Farkas [26, 27, 28] and others to give better bounds on the Kodaira dimension of $\mathcal{M}_g$, in particular to show that $\mathcal{M}_{23}$ is of general type.

In this chapter, we summarize techniques and the work of Green–Lazarsfeld, Donagi–Morrison, and Lelli-Chiesa [21, 32, 56, 57] on the question of extending linear systems from a curve to every curve in its linear system. In Section 4.2 we give a statement of the Donagi–Morrison conjecture, Conjecture 4.6, and results by Donagi–

Morrison and Lelli-Chiesa. We then highlight Lazarsfeld–Mukai bundle techniques used to prove cases of Conjecture 4.6. We conclude with a discussion of when Conjecture 4.6 fails and give a new bounded version.

Throughout this chapter, let S be a K3 surface. Unless otherwise specified, we will generally assume $C \subset S$ is a smooth curve.

Section 4.1
Constancy of the Clifford index in linear systems on K3 surfaces

In this section we give a brief historical account of the study of special linear systems on curves on K3 surfaces.

One could wish that if a curve $C \subset S$ has a g^r_d, then every curve in its linear system would also have a g^r_d. In fact, this was conjectured by Harris and Mumford for g^1_d's. A counterexample by Donagi [32] showed that this is not the case if S is a genus 2 K3 surface, i.e. a double cover of $\mathbb{P}^2$ branched along a smooth sextic. The example is written in detail in [21, Section 2.2], where Donagi and Morrison show that that if $\pi : S \to \mathbb{P}^2$ is the double cover and $E \subset \mathbb{P}^2$ is a non-singular cubic, then $C = \pi^{-1}(E)$ has a g^1_4 (since E has a g^1_2). However, the general smooth curve $C' \subset |C|$ has no g^1_4's, but does have a unique g^2_6. In fact, the general smooth curve $C' \in |C|$ is isomorphic to a plane sextic and thus has gonality 5. Later, Ciliberto–Pareschi [15] proved that this is the only counterexample to curves having non-constant gonality in linear systems on K3 surfaces with ample polarization, Knutsen later relaxed this to only globally generated polarization [49]. Thus in general one cannot hope that the gonality remains constant in a linear system.

Notice however, that $\gamma(g^1_4) = 2 = \gamma(g^2_6)$, and the Clifford index of curves in $|C|$ remains constant. This motivated Green to modify the conjecture to the constancy of the Clifford index in linear systems on K3 surfaces. This was proved by Green–Lazarsfeld using Lazarsfeld–Mukai bundles on K3 surfaces. Let us recall Theorem 3.4.

Theorem 4.1 (Green-Lazarsfeld [32])**.** *Let S be a K3 surface and $C \subset S$ a smooth irreducible curve of genus $g \geq 2$. Then $\gamma(C') = \gamma(C)$ for any smooth curve $C' \in |C|$. Moreover, if $\gamma(C) < \lfloor \frac{g-1}{2} \rfloor$, then there is a line bundle $L \in \mathrm{Pic}(S)$ such that $L|_{C'}$ computes $\gamma(C')$.*

The main technical aspect of the proof is to prove that there is non-trivial line bundle $N \subset E$, where E is a *reduction* of the Lazarsfeld–Mukai bundle $E_{C,A}$. We will not give details on what a reduction is, except to say that the reduction E behaves numerically and cohomologically as E_{C_1,A_1} for some curve $C_1 \in |C|$ and $A_1 \in \mathrm{Pic}(C_1)$ such that $\gamma(A_1) \leq \gamma(A)$ and $r(A_1) \leq r(A)$. In particular, $\det E = \det E_{C,A} = \mathcal{O}_S(C)$.

Once a line bundle $N \subset E$ is obtained, one checks that $h^0(C, N|_C), h^1(C, N|_C) \geq 2$, and so $N|_C$ contributes to the Clifford index $\gamma(C)$ of C. Finally, it remains to show that $\gamma(N|_C) \leq \gamma(C)$. In particular, one can ask whether N (or as we'll see $\det(E/N)$) gives a linear system on every smooth curve $C' \in |C|$ which computes the Clifford index of C'.

Another example by Donagi and Morrison shows that there may be g^r_d's which do propogate to curves $C' \in |C|$, but that are not of the form $L|_{C'}$ for a line bundle $L \in \mathrm{Pic}(S)$. Taking the double plane K3 surface again, taking $H = \pi^*\mathcal{O}_{\mathbb{P}^2}(4)$, Donagi and Morrison show that generic curves $C \in |H|$ have four g^1_6's which are not contained in the restriction of a linear system, and a line bundle of type g^2_8 that is the restriction of a line bundle on the K3 surface. They observe that in this case, the g^1_6's are contained in a g^2_8, that is on the curve C', each effective divisor in the linear system of the g^1_4's is contained in an effective divisor in the linear system of the g^2_8. Donagi and Morrison conjecture that this is a general phenomenon, which we discuss in the next section.

Section 4.2

Donagi–Morrison conjecture

Donagi and Morrison conjectured that linear systems on a curve on a K3 surface should extend to every linearly equivalent curve, or at least be contained in a linear system that is induced from the K3. That is, they conjectured that their counterexamples are the worst that can happen.

We require a few definitions.

Definition 4.2. We say $|A|$ is contained in the restriction of $|M|$ to C, if for every $A_0 \in |A|$, there is a divisor $M_0 \in |M|$ such that $A_0 \subset M_0 \cap C$. We note that if $H^0(S, M) \cong H^0(C, M|_C)$, then this is equivalent to there being a non-trivial map $A \to M|_C$ as sheaves on C (i.e. $h^0(C, A^\vee \otimes M|_C) > 0$).

Definition 4.3. A line bundle $M \in \mathrm{Pic}(S)$ is *adapted to* $|C|$ if

(i) $h^0(S, M) \geq 2$ and $h^0(S, \mathcal{O}_S(C) \otimes M^\vee) \geq 2$, and

(ii) $h^0(C', M|_{C'})$ is independent of the smooth curve $C' \in |C|$. In particular, $\gamma(M|_{C'})$ is independent of C'.

Lemma 4.4 ([21, Lemma 5.2])**.** *M is adapted to $|C|$ if*

(i) $h^0(S, M) \geq 2$ and $h^0(S, \mathcal{O}_S(C) \otimes M^\vee) \geq 2$, and

(ii) either $h^1(S, M) = 0$ or $h^1(S, \mathcal{O}_S(C) \otimes M^\vee) = 0$.

Conjecture 4.5 (Original Donagi–Morrison Conjecture [21, Conjecture 1.2])**.** *Let S be a K3 surface, C a smooth curve on S with genus $g \geq 2$, and A a complete basepoint free g^r_d on C such that $r \geq 1$, $d \leq g-1$, and $\rho(C, A) = \rho(g, r, d) < 0$. Then there exists a line bundle $M \in \mathrm{Pic}(S)$ such that $|A|$ is contained in the restriction of $|M|$ to C with*

$$c_1(M).C \leq g - 1 \text{ and } \gamma(M|_C) \leq \gamma(A).$$

Conjecture 4.5, however, turns out to be false in general. Indeed, Lelli-Chiesa gives a counterexample, again with S a double cover of $\mathbb{P}^2$ branches along a smooth sextic. In [57, Counterexample 1], Lelli-Chiesa shows that the condition $c_1(M).C \leq g-1$ can be violated, however, the counterexample does not preclude the existence of a line bundle $M \in \mathrm{Pic}(S)$ such that $|A|$ is contained in the restriction of $|A|$ to C. Thus, Lelli-Chiesa suggests replacing this condition with the requirement that M is adapted to $|C|$.

Conjecture 4.6 (Donagi–Morrison Conjecture, [57] Conjecture 1.3)**.** *Let (S, H) be a polarized K3 surface and $C \in |H|$ be a smooth irreducible curve of genus ≥ 2. Suppose A is a complete basepoint free g^r_d on C such that $d \leq g-1$ and $\rho(g, r, d) < 0$. Then there exists a line bundle $M \in \mathrm{Pic}(S)$ adapted to $|H|$ such that $|A|$ is contained in the restriction of $|M|$ to C and $\gamma(M|_C) \leq \gamma(A)$.*

Definition 4.7. Let (S, H) be a polarized K3 surface and $C \in |H|$ be a smooth irreducible curve of genus ≥ 2. Suppose A is a complete basepoint free g^r_d on C such that $d \leq g-1$ and $\rho(g, r, d) < 0$. We call a line bundle $M \in \mathrm{Pic}(S)$ a *Donagi–Morrison lift* of A if M satisfies the conditions in Conjecture 4.6: namely,

- M is adapted to $|H|$,
- $|A|$ is contained in the restriction of $|M|$ to C, and
- $\gamma(M|_C) \leq \gamma(A)$.

We call a line bundle M a *potential Donagi–Morrison lift* of A if M satisfies $\gamma(M|_C) \leq \gamma(A)$ and $d(M|_C) \geq d(A)$. Note that a Donagi–Morrison lift is a potential Donagi–Morrison lift. We say a (potential) Donagi–Morrison lift is of type g^s_e if $M^2 = 2s-2$ and $M.H = e$.

Donagi and Morrison prove Conjecture 4.6 in the case that $r = 1$, extending results of Reid [75].

Theorem 4.8 ([21, Theorem 5.1']). *Let C be a smooth non-hyperelliptic curve on a K3 surface S and suppose A is a complete basepoint free g^1_d on C with $\rho(C, A) < 0$. Then there is a line bundle $M \in \mathrm{Pic}(S)$ adapted to $|C|$ such that*

- $\gamma(M|_C) \leq \gamma(A)$.
- *$|A|$ is contained in the restriction of $|M|$ to C.*

Following this, Lelli-Chiesa proved Conjecture 4.6 in the case $r = 2$ under mild hypotheses on the curve C. We discuss this result in more detail in Section 4.4.

Theorem 4.9. *Let S be a K3 surface and $H \in \mathrm{Pic}(S)$ be an ample line bundle such that a general curve $C \in |H|$ has genus g, Clifford dimension 1 and maximal Clifford index $\gamma(C) = \frac{g-1}{2}$. Let A be a complete basepoint free g^2_d on a smooth irreducible curve $C \in |H|$ such that $\rho(C, A) < 0$. Then Conjecture 4.6 holds for A. Moreover, one has $c_1(M).C \leq \frac{4g-4}{3}$.*

We prove in [6], also Section 5.2, a bounded version of Conjecture 4.6 holds in the case $r = 3$. Currently, there are no known general results concerning Conjecture 4.6 when $r \geq 4$.

However, if we add an additional assumption, namely that we have control over the Clifford index of the line bundle, then Lelli-Chiesa's theorem extends Green and Lazarsfeld's theorem.

Theorem 4.10 ([57, Theorem 4.2]). *Let A be a complete g^r_d on a non-hyperelliptic and non-trigonal curve $C \subset S$ such that $d \leq g - 1$, $\rho(g, r, d) < 0$ and $\gamma(A) = \gamma(C)$. Assume $H := \mathcal{O}_S(C)$ is ample and the following condition is satisfied:*

(∗) *there is no irreducible elliptic curve $\Sigma \subset S$ such that $\Sigma \cdot C = 4$ and no irreducible genus 2 curve $B \subset S$ such that $B \cdot C = 6$.*

Then, one of the following occurs:

(i) There exists a line bundle $M \in \mathrm{Pic}(S)$ adapted to $|H|$ such that $A \simeq M|_C$.

(ii) The line bundle A satisfies $h^0(C, A) = 2$ (i.e., $r = 1$); furthermore, there exists a line bundle $M \in \mathrm{Pic}(S)$ adapted to $|H|$ such that $|A|$ is contained in the restriction of $|M|$ to C.

If condition (*) is not satisfied, then the following cases may also occur:

(iii) There exists an irreducible curve B of genus 2 such that $C \sim 3B$ and A is either a complete g^2_6 or a complete g^3_8; in both cases $|A|$ is contained in $|\mathcal{O}_C(2B)|$.

(iv) There exist an irreducible curve B of genus 2 and an irreducible elliptic curve Σ such that $B \cdot \Sigma = 2$ and $C \sim 2B + \Sigma$; furthermore, A is of type either g^2_6 or g^3_8 and $|A|$ is contained in $|\mathcal{O}_C(B + \Sigma)|$.

(v) One has $C \sim \Sigma + \Sigma' + \Sigma''$ for three irreducible elliptic curves Σ, Σ', Σ'' pairwise intersecting in two points, and A is of type g^2_6; moreover, the linear system $|A|$ is contained in $|\mathcal{O}_C(\Sigma + \Sigma'')|$.

(vi) There exist two irreducible elliptic curves Σ, Σ' and a divisor D on S satisfying $D^2 = -4$, and $D \cdot \Sigma' = 0$, and $D \cdot \Sigma = \Sigma \cdot \Sigma' = 2$ such that $C \sim D + 2\Sigma + \Sigma'$; furthermore, A is of type g^2_6 and $|A|$ is contained in $|\mathcal{O}_C(\Sigma + \Sigma')|$.

(vii) There are two elliptic curves Σ, Σ' and a divisor D such that $C \sim D + 2\Sigma + \Sigma'$ with $D^2 = -2$, and $D \cdot \Sigma' = 0$, and $D \cdot \Sigma = \Sigma \cdot \Sigma' = 2$; furthermore, A is of type g^2_6 and $|A|$ is contained in $|\mathcal{O}_C(\Sigma + \Sigma')|$.

(viii) There is an irreducible elliptic curve Σ and a divisor D such that $C \sim 2D + 3\Sigma$, and $D^2 = -2$ and $D \cdot \Sigma = 2$; moreover, A is a complete g^3_8 and $|A|$ is contained in $|\mathcal{O}_C(D + 2\Sigma)|$.

(ix) There are two irreducible elliptic curves Σ, Σ' satisfying $\Sigma \cdot \Sigma' = 2$ and $C \sim 2\Sigma + 2\Sigma'$; moreover, either A is of type g^2_6 and $|A|$ is contained in $|\mathcal{O}_C(\Sigma + \Sigma')|$, or A is of type g^3_8 and $|A|$ is contained in $|\mathcal{O}_C(2\Sigma + \Sigma')|$.

Notice that in all the cases (iii)–(ix) one has $\gamma(C) = 2$ and $g \leq 10$. In fact, condition $(*)$ is automatically satisfied as soon as $\gamma(C) > 2$.

Section 4.3

Lazarsfeld–Mukai bundles and lifting

We highlight how Lazarsfeld–Mukai bundles are used in proofs of specific cases of Conjecture 4.6.

A crucial technique in the proof of Theorem 4.10 is similar to the technique used by Green–Lazarsfeld. Namely, Lelli-Chiesa shows that one needs to find a nontrivial line bundle in the Lazarsfeld–Mukai bundle.

Proposition 4.11 ([57, Proposition 5.1])**.** *Under the hypotheses of Conjecture 4.6, let A be primitive and assume the existence of a globally generated line bundle $N \in \mathrm{Pic}(S)$ such that N is a saturated subsheaf of $E_{C,A}$.*

Then, Conjecture 4.6 holds with $M := H \otimes N^\vee$ if $h^1(S, N) = 0$, and with $M := H(-\Sigma)$ if $c_1(N) = k\Sigma$ for an irreducible elliptic curve $\Sigma \subset S$ and an integer $k \geq 2$.

The trouble is, however, that obtaining the line bundle N can be difficult.

The proofs of the results of Donagi–Morrison and Lelli-Chiesa use Lazarsfeld–Mukai bundles $E_{C,A}$ associated to the pair (C, A), and the fact that when the vector bundle $E_{C,A}$ has a nontrivial maximal destabalizing sub-line bundle $N \in \mathrm{Pic}(S)$, then $|A|$ is contained in the restriction of $|H \otimes N^\vee|$. For rank 2 linear systems, a case-by-case analysis of the Jordan–Hölder and Harder–Narasimhan filtrations of $E_{C,A}$ is used. This technique becomes much more difficult in higher rank.

We recall [57, Lemma 4.1] and fill in details for the proof, which shows when a linear series on a curve $C \in |H|$ is the restriction of a line bundle L on S.

Lemma 4.12 ([57] Lemma 4.1). *Let $N_1 \in \mathrm{Pic}(S)$ satisfy $h^0(S, N_1) \geq 2$ and $h^1(S, N_1) = 0$. Also assume that the line bundle $N_2 := H \otimes N_1^\vee$ is globally generated and $h^1(S, N_2) = 0$. Let E be a gLM bundle on S. Then, $E = E_{C, N_2|_C}$ for some smooth irreducible curve $C \in |H|$ if and only if E is an extension of the form*

$$0 \longrightarrow N_1 \longrightarrow E \longrightarrow E_{D,\omega_D} \longrightarrow 0$$

for some smooth irreducible curve $D \in |N_2|$.

Proof. Suppose first that $E = E_{C,N_2|_C}$ for a smooth irreducible curve $C \in |H|$. Since N_2 is globally generated and $h^1(S, N_1) = h^1(S, N_1^\vee) = 0$ we have $H^0(S, N_2) = H^0(S, N_2|_C)$. Thus we have an evaluation map

$$H^0(C, N_2|_C) \otimes \mathcal{O}_S = H^0(S, N_2) \to N_2.$$

We obtain a diagram where the top sequence is the definition of the LM bundle $E^\vee_{C,N_2|_C}$ and the bottom sequence comes from tensoring the sequence for the divisor C with N_2:

$$\begin{tikzcd}
0 \arrow[r] & E^\vee_{C,N_2|_C} \arrow[r] \arrow[d, dotted] & H^0(C, N_2|_C) \otimes \mathcal{O}_S \arrow[r] \arrow[d] & N_2|_C \arrow[r] \arrow[d, equal] & 0 . \\
0 \arrow[r] & N_1^\vee \arrow[r] & N_2 \arrow[r] & N_2|_C \arrow[r] & 0
\end{tikzcd}$$

Thus there is a map $E^\vee_{C,N_2|_C} \to N_1^\vee$ making the above diagram commute. Letting $K^\vee$ be the kernel of the evaluation map $H^0(C, N_2|_C) \otimes \mathcal{O}_S = H^0(S, N_2) \to N_2$, and κ the cokernel of the evaluation map (which in this case is zero as $H^1(S, N_2) = 0$), we obtain an enlarged diagram where equality of the kernels and cokernels follows from the snake lemma

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & K^\vee \arrow[r, equal] \arrow[d] & K^\vee \arrow[d] & & \\
0 \arrow[r] & E^\vee_{C,N_2|_C} \arrow[r] \arrow[d, dotted] & H^0(C, N_2|_C) \otimes \mathcal{O}_S \arrow[r] \arrow[d] & N_2|_C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N_1^\vee \arrow[r] \arrow[d] & N_2 \arrow[r] \arrow[d] & N_2|_C \arrow[r] & 0 \\
 & \kappa \arrow[r, equal] \arrow[d] & \kappa \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}.$$

As $\kappa = 0$, we can verify that K is a globally generated vector bundle satisfying $h^1(S, K) = h^2(S, K) = 0$, and thus K is a LM bundle. Moreover, from the left vertical short exact sequence one finds $c_1(K) = c_1(N_2)$ and from the middle vertical sequence finds $c_2(K) = c_1(N_2)^2$. Since $h^1(S, N_1^\vee) = h^1(S, N_2 - H) = 0$, Lemma 4.14 shows that $c_1(N_2)^2 = 2(\mathrm{rk}(E) - 2) = 2(\mathrm{rk}(K) - 1)$. Thus $\mathrm{Cliff}(K) = 0$ and Lemma 3.21 yields $K \simeq E_{D,\omega_D}$ for a smooth irreducible curve $D \in |N_2|$. Thus the forward direction is proved.

For the other direction, let E be an extension as above. Then, E is locally free and globally generated as N_1 is globally generated, and E_{D,ω_D} is either globally generated or at least globally generated off a finite set. Furthermore, via the long exact sequence in cohomology, we see $h^1(S, E) = h^2(S, E) = 0$. Thus E is a gLM bundle of types (I) and (II), hence is of the form $E_{C,A}$ for a smooth curve $C \in |\det(E)| = |H|$ and a line bundle $A \in \mathrm{Pic}(C)$ with $d = c_2(E)$. Hence E sits in a diagram

$$\begin{tikzcd} 0 \arrow[r] & H^0(C,A)^\vee \otimes \mathcal{O}_S \arrow[r] & E \arrow[r, "\alpha"] & \iota_*(\omega_C \otimes A^\vee) \arrow[r] & 0. \\ & & N_1 \arrow[u, hook, "\phi"] & & \end{tikzcd}$$

Since $h^0(N_1) \geq 2$, we have $\mathrm{Hom}(N_1, \mathcal{O}_S) = 0$: as otherwise by Serre duality $h^0(S, N_1^\vee) \neq 0$, thus N_1 would be a trivial line bundle with $h^0(S, N_1) = 1$. Hence we also have

$$0 \neq \alpha \circ \phi \in \mathrm{Hom}(N_1, \iota_*(\omega_C \otimes A^\vee))$$

as otherwise ϕ would factor through $H^0(C,A)^\vee \otimes \mathcal{O}_S$. We thus have

$$\mathrm{Hom}(N_1, \iota_*(\omega_C \otimes A^\vee)) = H^0(S, \iota_*(A^\vee) \otimes N_2|_C) \neq 0,$$

as $\omega_C = H|_C$ by adjunction. Hence $|A|$ is contained in $|N_2|_C|$. By adjunction we have $\omega_D = N_2|_D$, and so $N_2^2 = \deg_D(\omega_D) = c_2(E_{D,\omega_D})$. Applying Whitney's formula to the original extension, we find

$$\begin{aligned} 1 + H + d &= (1 + N_1)\left(1 + H - N_1 + N_2^2\right) \\ &= (1 + N_1)\,(1 + H - N + (H - N_1)(N_2)) \\ &= (1 + N_1)\,(1 + H - N + H.N_2 - N_1(H - N_1)) \\ &= 1 + N_1 + H - N_1 + H.N_1 - N_1^2 + H.N_2 - H.N_1 + N_1^2 \\ &= 1 + H + H.N_2 \end{aligned}$$

It follows that $\deg(N_2|_C) = H.N = d$. We see that $A^\vee \otimes N_2|_C$ is effective as it has a global section and moreover has degree 0 on C, and hence is the trivial bundle. Whence A is isomorphic to $N_2|_C$. □

Remark 4.13 ([57] Remark 6). The above proof shows that as soon as we have a nontrivial $N \in \mathrm{Pic}(S)$ with $h^0(S, N) \neq 0$ and $N \hookrightarrow E_{C,A}$, we have

$$h^0(S, \iota_*(A) \otimes (H \otimes N^\vee)|_C) = h^0(C, A^\vee \otimes (H \otimes N^\vee)|_C) \neq 0,$$

i.e., the linear series $|A|$ is contained in $|(H \otimes N^\vee)|_C|$. We also note that if $h^1(S, N) = 0$, then

$$H^0(C, (H \otimes N^\vee)|_C) = H^0(S, H \otimes N^\vee)|_C.$$

We recall a few lemmas which show when a linear series on a curve $C \in |H|$ is the restriction of a line bundle L on S.

Lemma 4.14. *Let (S, H) be a polarized K3 surface of genus $g \geq 2$, $C \in |H|$ be a smooth irreducible curve, and L a globally generated line bundle on S such that $L|_C$ is a g^r_d with $c_1(L).C = d < 2g - 2$. Then if $h^1(S, L) = 0$, we have $L^2 = 2r - 2 - 2h^1(S, L(-C))$.*

Proof. Since H is basepoint free and $c_1(L(-C)).C = d - (2g - 2) < 0$, we have $h^0(S, L(-C)) = 0$, as in the proof of [48, Proposition 2.1]. We now consider the short exact sequence for a divisor $C \subset S$ tensored with L,

$$0 \longrightarrow L(-C) \longrightarrow L \longrightarrow L|_C \longrightarrow 0.$$

By Riemann–Roch on C we have $h^1(S, L|_C) = h^1(C, L|_C) = r - d + g$, and as $h^1(S, L) = h^2(S, L) = 0$, the long exact sequence in cohomology and Serre duality give $h^2(S, L(-C)) = h^0(S, L(-C)^\vee) = r - d + g$. By Riemann–Roch on S, we have

$$\begin{aligned} h^0(S, L(-C)^\vee) - h^1(S, L(-C)) &= 2 + \frac{c_1(L(-C))^2}{2} \\ &= 2 + \frac{c_1(L)^2 - 2d + 2g - 2}{2} \\ &= 1 - d + g + \frac{c_1(L)^2}{2} \end{aligned}$$

thus $c_1(L)^2 = 2r - 2 - 2h^1(S, L(-C))$. □

Corollary 4.15. *Let (S, H) be a polarized K3 surface of genus $g \geq 2$, A a complete g^r_d on a smooth $C \in |H|$. Let $N \in \mathrm{Pic}(S)$ be a line bundle with $h^0(S, N) \geq 2$ and $h^1(S, N) = 0$. Assume $H \otimes N^\vee$ is globally generated, satisfies $h^1(S, H \otimes N^\vee) = 0$, and $(H \otimes N^\vee)|_C \cong A$. Then $c_1(H \otimes N^\vee)^2 = 2r - 2$.*

Proof. We have $h^1(S, N) = 0$. Hence as $N^\vee = H \otimes N^\vee \otimes H^\vee$, Serre duality gives $0 = h^1(S, N^\vee) = h^1(S, H \otimes N^\vee(-C))$. Thus Lemma 4.14 shows that $(H - N)^2 = 2r - 2$. □

Lemma 4.16. *Let N be a line bundle and $0 \to N \to E \to E/N \to 0$ be a short exact sequence of coherent sheaves on a polarized K3 surface (S, H), where E/N is stable, $\mathrm{rk}(E) = r + 1$, $c_1(E) = H$, $c_1(E)^2 = 2g - 2 \geq 0$. If $h^0(S, N) < 2$, then $c_2(E) \geq \frac{g(r-1)}{r} + \frac{2g-2}{r(r+1)} + r - \frac{1}{r}$.*

Proof. Since $\mu(N) \geq \mu(E) \geq 0$, we have $h^2(S, N) = 0$. Therefore if $h^0(S, N) < 2$ we have $c_1(N)^2 \leq -2$. Hence

$$c_1(E/N)^2 + 2c_1(N).c_1(E/N) = c_1(E)^2 - c_1(N)^2 \geq 2g - 2 + 2 = 2g$$

and

$$c_1(E/N).c_1(N) = c_1(N).(c_1(E) - c_1(N)) \geq \frac{2g-2}{r+1} + 2,$$

where the last inequality comes from the fact that $\mu(N) \geq \mu(E)$. Thus $\frac{c_1(E/N)^2}{2} \geq g - c_1(N).c_1(E/N)$.

Furthermore, since E/N is stable of rank r, the dimension of the moduli space of stable sheaves with Mukai vector $\nu(E/N)$, $M^s_{\nu(E/N)}$, has non-negative dimension. Thus $2rc_2(E/N) - (r-1)c_1(E/N)^2 - 2(r^2-1) \geq 0$, and we have $c_2(E/N) \geq r - \frac{1}{r} + \left(\frac{r-1}{2r}\right) c_1(E/N)^2$.

We now calculate $c_2(E) = c_1(E/N).c_1(N) + c_2(E/N) \geq \frac{g(r-1)}{r} + \frac{2g-2}{r(r+1)} + r - \frac{1}{r}$, as desired. ☐

We present a version of [56, Proposition 7.4] which motivates our proof strategy for Conjecture 4.6 when $r = 3$.

Proposition 4.17. *Let (S, H) be a polarized K3 surface and A be a complete basepoint free g^r_d on a smooth irreducible curve $C \in |H|$ with $r \geq 2$ and let $E = E_{C,A}$. Suppose that E sits in a short exact sequence*

$$0 \longrightarrow N \longrightarrow E \longrightarrow E/N \longrightarrow 0$$

for some line bundle N and $c_2(E) = d < \frac{g(r-1)}{r} + \frac{2g-2}{r(r+1)} + r - \frac{1}{r}$. If E/N is stable, or E/N is semistable and there are no elliptic curves on S, then $|A|$ is contained in the restriction to C of the linear system $|H \otimes N^\vee|$ on S. Moreover, $H \otimes N^\vee$ is adapted to $|H|$ and $\gamma(H \otimes N^\vee|_C) \leq d - r - 3$.

Proof. By the previous lemma, $h^0(S, N) \geq 2$. We also have $h^0(S, \det E/N) \geq 2$ from [56, Lemma 3.3]. We note that $(E/N)^{\vee\vee}$ is globally generated off a finite set and

$$h^i(S, (E/N)^{\vee\vee}) = h^i(S, E/N) = 0 \text{ for } i = 1, 2.$$

Since $\det E/N = \det(E/N)^{\vee\vee}$ is basepoint free and nontrivial, $\det E/N$ is nef, thus

$c_1(E/N)^2 \geq 0$. If $h^1(S, \det E/N) \neq 0$, then $c_1(E/N)^2 = 0$ by Saint-Donat. By [32, Proposition 1.1], there is a smooth elliptic curve $\Sigma \subset S$ such that $(E/N)^{\vee\vee} = \mathcal{O}(\Sigma)^{\oplus 3}$. This contradicts the stability of E/N (or the non-existence of elliptic curves on S), thus we must have $c_1(E/N)^2 \geq 2$ (hence $c_2(E/N) \geq r+1$) and $h^1(S, \det E/N) = 0$. This ensures that $h^0(C, \det E/N|_C) = h^0(C, H \otimes N^\vee|_C)$ does not depend on the curve $C \in |H|_s$. Hence $\det E/N = H \otimes N^\vee$ is adapted to $|H|$. We calculate

$$\begin{aligned}
\gamma(\det E/N|_C) &= c_1(E/N).c_1(E) - 2h^0(C, \det E/N|_C) + 2 \\
&= c_1(E/N)^2 + c_1(N).c_1(E/N) - 2h^0(C, \det E/N|_C) + 2 \\
&\leq c_1(E/N)^2 - 2h^0(S, \det E/N) + c_1(N).c_1(E/N) + 2 \\
&= -2h^1(S, \det E/N) - 4 + c_1(N).c_1(E/N) + 2 \\
&= d - c_2(E/N) - 2 \leq d - r - 3.
\end{aligned}$$

The claim that $|A|$ is contained in $|H \otimes N^\vee|_C|$ is proved in the same way as in [57, Lemma 4.1]. □

Remark 4.18. In the above proposition, if A is of type g^3_d, then $\gamma(H \otimes N^\vee|_C) \leq d - r - 3 = \gamma(A)$. However, as soon as $r \geq 4$, then $\gamma(H \otimes N^\vee|_C)$ may be bigger than $\gamma(A)$.

Section 4.4

Bounded Donagi–Morrison conjecture

In this section, we outline a further modification of Conjecture 4.6 that is already implicit in Lelli-Chiesa's theorem verifying Conjecture 4.6 in the case $r = 2$.

The proofs of cases of Conjecture 4.6 have all used the idea of finding a maximal destabilizing subline bundle of $E_{C,A}$, that is Proposition 4.17. Thus one can ask whether a stronger version of Conjecture 4.6 holds.

Conjecture 4.19 (Strong Donagi–Morrison Conjecture). *Let (S, H) be a polarized K3 surface and A be a complete basepoint free g^r_d on a smooth irreducible curve $C \in |H|$ with $r \geq 2$. Then there is a nontrivial line bundle $N \hookrightarrow E_{C,A}$ with $h^0(S, N) \geq 2$ such that $E_{C,A}/N$ is stable.*

As stated, this conjecture is false, see [57, Appendix A]. We give the details for one example.

Example 4.20. In [57, Appendix A, Remark 12], Knutsen and Lelli-Chiesa construct examples of K3 surfaces S of Picard rank 2 such that a smooth irreducible curve $C \subset S$ has a Brill–Noether special linear system A of rank 3 with $\rho(A) = -1$ whose Lazarseld–Mukai bundle $E_{C,A}$ admits no effective sub-line bundle. That is, Proposition 4.17 cannot be used to find a Donagi–Morrison lift of A. Here, we give an explicit example and explain how it relates to our results.

We first recall Knutsen and Lelli-Chiesa's construction. For even integers $a, b \geq 4$ and $d = a+b$, let S be a K3 surface with $\mathrm{Pic}(S) = \Lambda^b_{a,d}$. That is, $\mathrm{Pic}(S) = \mathbb{Z}[H] \oplus \mathbb{Z}[D]$ with $H^2 = 2a - 2 \geq 4$, $D^2 = 2b - 2 \geq 4$, $H.D = d$, with a, b even and $d = a + b$. Suppose that $\mathrm{Pic}(S)$ has no classes of self-intersection -2 or 0. There are infinitely many choices of a and b that satisfy these hypotheses, and such that every element of the linear systems $|H|$ and $|L|$ are reduced and irreducible; these are examples of the so-called *Knutsen K3 surfaces* in [2]. Thus general curves $C_1 \in |H|$ and $C_2 \in |L|$ are smooth of genus a and b, and by Lazarsfeld's theorem [53], are Brill–Noether general, in particular, have generic gonality $k_1 = (a + 2)/2$ and $k_2 = (b + 2)/2$, respectively. Let E_1 and E_2 be the LM bundles associated to gonality pencils $g^1_{k_1}$ on C_1 and a $g^1_{k_2}$ on C_2. As these pencils are Brill–Noether general, the LM bundles E_1 and E_2 are simple, hence admit no injective map from an effective line bundle N. A calculation using Remark 3.16 shows that the vector bundle $E = E_1 \oplus E_2$ is a LM bundle associated to a linear system A of type $g^3_{k_1+k_2+d}$ on a smooth irreducible curve $C \in |H+L|$. We note that $g(C) = 2d-1$, and that $\rho(A) = -1$. However, since E admits no injective map $N \hookrightarrow E$, the linear system A admits no Donagi–Morrison lift, and so Conjecture 4.6 and even Conjecture 4.19 fail for (C, A).

The first case where such an example shows the failure of Conjecture 4.6 for (C, A) is genus 19, with $a = 6$ and $b = 4$. The corresponding polarized K3 surface $(S, H+L)$ of genus 19 has $\mathrm{Pic}(S) = \Lambda^4_{19,16}$ with basis $H+L, L$. In the proof of Proposition 6.38, we needed the Donagi–Morrison Conjecture (Conjecture 4.6) for linear systems on curves on a different lattice polarized K3 surface, showing that our bounded version (Theorem 5.15) is in some sense tight (at least in genus 19).

However, a bounded version as in [6, 56, 57, 75] may be reasonable.

Conjecture 4.21 (Bounded Strong Donagi–Morrison Conjecture)**.** *Let (S, H) be a polarized K3 surface and A be a complete basepoint free g^r_d on a smooth irreducible curve $C \in |H|$ with $r \geq 2$. Then there is a bound $\beta(C, S)$ depending on C and S such that if $d < \beta(C, S)$, then there is a line bundle $N \hookrightarrow E_{C,A}$ with $h^0(S, N) \geq 2$ such that $E_{C,A}/N$ is stable.*

This is what we prove in the case $r = 3$ in [6] and Chapter 5.

Chapter 5

Rank 4 Lazarsfeld–Mukai bundles

In Section 5.1, we first reduce the problem of verifying Conjecture 4.6 in the case $r = 3$ to finding a bound for each terminal filtration of the Lazarsfeld–Mukai bundle associated to the g^3_d, a filtration obtained by taking the Harder–Narasimhan and Jordan–Hölder filtrations of the Lazarsfeld–Mukai bundle. We then find a bound on the degree of the g^3_d for each filtration. In Section 5.2, after having obtained bounds for every terminal filtration that does not have a maximal destabilizing sub-line bundle, we give the proof of Conjecture 4.21 when $r = 3$. These sections are taken from [6].

Section 5.1 Filtrations of Lazarsfeld–Mukai Bundles of Rank 4

Throughout this chapter, (S, H) is a polarized K3 surface of genus g, $C \in |H|$ is a smooth irreducible curve, A is a line bundle of type g^3_d on C, and $E = E_{C,A}$ is the LM bundle corresponding to A. Given E, we can take its JH filtration or take its HN filtration, further take JH filtrations of the properly semistable factors, lift the JH factors and expand the HN filtration of E to arrive at a *terminal filtration* such that all quotients are stable sheaves. We enumerate all the possibilities listing a filtration by the ranks of the terms, i.e., a filtration of type $1 \subset 4$ is a filtration $0 \subset N \subset E$ where $\mathrm{rk}(N) = 1$.

The terminal filtrations correspond to flags of E where each quotient is stable, hence the terminal filtrations are

$$1 \subset 4, \quad 2 \subset 4, \quad 3 \subset 4,$$
$$1 \subset 2 \subset 4, \quad 1 \subset 3 \subset 4, \quad 2 \subset 3 \subset 4,$$
$$1 \subset 2 \subset 3 \subset 4.$$

In order to apply Proposition 4.17, we want to show that given the g^3_d, E must have a terminal filtration of type $1 \subset 4$. In all other cases, we want to find a lower bound on $d = c_2(E)$. To this end, we find a bound for $c_2(E)$ in terms of the intersections of the Chern roots of the LM bundle E. We begin by noting a few general bounds, and then deal with each filtration.

We slightly generalize the proof of [56, Lemma 4.1].

Proposition 5.1. *Let E a LM bundle with $c_1(E) = H$ and $\mu(E) = \frac{g-1}{2} > 0$ sitting in an exact sequence*

$$0 \longrightarrow M \longrightarrow E \longrightarrow M_1 \longrightarrow 0$$

where M and M_1 are coherent sheaves. Suppose that the general smooth curve $C \in |H|$ has (constant) Clifford index $\gamma = \gamma(C)$. Then one has $c_1(M).c_1(M_1) \geq \gamma + 2$.

Proof. We write $\mu(F) = \mu_H(F)$. Since M_1 is a quotient of E, it is globally generated off a finite set of points. Moreover, we have $h^2(S, M_1) = 0$, thus $h^0(S, \det M_1) \geq 2$ by [56, Lemma 3.3] as the vector bundle $M_1^{\vee\vee}$ is globally generated off a finite number of points and $\det(M_1) := \det(M_1^{\vee\vee})$. As in [56, Lemma 3.2], we see that $\det M_1$ is basepoint free and nontrivial, thus $\mu(\det M_1) > 0$, $\mu(M) > 0$. Hence as $\mu(\det M) \geq \mu(M) > 0$, [56, Proposition 3.1] shows that $h^2(S, \det M_1) = 0$, $h^2(S, \det M) = 0$, and that $\det M_1$ is nef whereby $c_1(M_1)^2 \geq 0$.

Furthermore, as

$$\mu(M) = \frac{c_1(M).c_1(E)}{\mathrm{rk}(M)} = \frac{c_1(M).(c_1(M) + c_1(M_1))}{\mathrm{rk}(M)} \geq \frac{g-1}{2},$$

we have $c_1(M).c_1(M_1) \geq \mathrm{rk}(M)\frac{g-1}{2} - c_1(M)^2$. Since $h^2(S, \det M) = 0$, we note that

$$h^0(S, \det M) \geq h^0(S, \det M) - h^1(S, \det M) = \chi(\det M) = 2 + \frac{c_1(M)^2}{2}.$$

Therefore, if $2 > h^0(S, \det M)$, then $c_1(M)^2 \leq -2$, and thus

$$c_1(M).c_1(M_1) \geq \mathrm{rk}(M)\frac{g-1}{2} + 2 \geq \mathrm{rk}(M)\gamma + 2 \geq \gamma + 2$$

as $\mathrm{rk}(M) \geq 1$.

Hence from now on we assume that $h^0(S, \det M) \geq 2$. Since $\omega_C \otimes (\det M_1)^\vee \otimes \mathcal{O}_C = \det M \otimes \mathcal{O}_C$, the line bundle $\det M_1 \otimes \mathcal{O}_C$ contributes to $\gamma(C)$. Tensoring the short exact sequence for $\mathcal{O}_C$ with $\det M_1$ gives

$$0 \longrightarrow \det M^\vee \longrightarrow \det M_1 \longrightarrow \det M_1 \otimes \mathcal{O}_C \longrightarrow 0\,,$$

which gives $h^0(C, \det M_1 \otimes \mathcal{O}_C) \geq h^0(S, \det M_1)$. It follows that

$$\begin{aligned}\gamma(\det M_1 \otimes \mathcal{O}_C) &= c_1(M_1).(c_1(M) + c_1(M_1)) - 2h^0(C, \det M_1 \otimes \mathcal{O}_C) + 2 \\ &\leq c_1(M_1)^2 + c_1(M).c_1(M_1) - 2\chi(\det M_1) - 2h^1(S, \det M_1) + 2 \\ &= -2 + c_1(M).c_1(M_1) - 2h^1(S, \det M_1).\end{aligned}$$

By assumption, we have $\gamma(\det M_1 \otimes \mathcal{O}_C) \geq \gamma$, thus

$$c_1(M).c_1(M_1) \geq \gamma + 2 + 2h^1(S, \det M_1) \geq \gamma + 2,$$

as desired. □

Remark 5.2. It follows from the second half of the proof that if M and M_1 are coherent sheaves such that $c_1(M)+c_1(M_1) = c_1(E)$, $\det M_1 \otimes \mathcal{O}_C$ (hence also $\det M \otimes \mathcal{O}_C$) contributes to $\gamma(C)$, and $h^2(S, \det M_1) = 0$ (or $h^2(S, \det M) = 0$), then $c_1(M).c_1(M_1) \geq \gamma(C)+2+2h^1(S, \det M_1) \geq \gamma(C)+2$ (or $c_1(M).c_1(M_1) \geq \gamma(C)+2+2h^1(S, \det M) \geq \gamma(C) + 2$).

Proposition 5.3. *Let (S, H) be a polarized K3 surface, $C \in |H|$ a smooth irreducible curve, A a basepoint free line bundle on A of type g^3_d, and $E = E_{C,A}$. Suppose E sits in an exact sequence*

$$0 \longrightarrow M \longrightarrow E \longrightarrow E/M \longrightarrow 0\,,$$

where M and E/M are coherent torsion free sheaves on S and $\mu(M) \geq \mu(E) \geq \mu(E/M)$. If $\mathrm{rk}(M) \geq \mathrm{rk}(E/M)$, then $c_1(M)^2 \geq c_1(E/M)^2$. And if $\mathrm{rk}(M) > \mathrm{rk}(E/M)$, then $c_1(M)^2 > c_1(E/M)^2$. In particular, $\det(E/M) \otimes \mathcal{O}_C$ contributes to $\gamma(C)$.

Proof. As in Proposition 5.1, we see $h^0(S, \det E/M) \geq 2$, $\mu(E/M) > 0$, $\det(E/M)$

is nef, and $h^2(S, \det M) = 0$. Since $h^0(S, \det E/M) \geq 2$, it remains to show that $h^0(S, \det M) \geq 2$.

We observe that

$$\begin{aligned} c_1(M)^2 + c_1(M).c_1(E/M) &= \mathrm{rk}(M)\mu(M) \\ &\geq \mathrm{rk}(E/M)\mu(E/M) = c_1(E/M)^2 + c_1(M).c_1(E.M) \end{aligned}$$

whence $c_1(M)^2 \geq c_1(E/M)^2 \geq 0$ as $\det(E/M)$ is nef.

Since $h^2(S, \det M) = 0$, we have $h^0(S, \det M) \geq \chi(\det M) = 2 + \frac{c_1(M)^2}{2}$. Thus as $c_1(M)^2 \geq 0$, $\det(E/M) \otimes \mathcal{O}_C$ contributes to $\gamma(C)$. □

For each terminal filtration not of the form $0 \subset 1 \subset 4$, we find a lower bound for $d = c_2(E)$. That is whenever E does not have a maximal destabilizing sub-line bundle, we find that d must be large. In effect, $c_2(E)$ controls the complexity of its Harder–Narasimhan and Jordan–Hölder filtrations.

5.1.1. Filtration $2 \subset 4$

We assume E is unstable with terminal filtration $0 \subset M \subset E$ with M and $M_1 = E/M$ stable rank 2 torsion free sheaves. Thus E sits in an exact sequence of the form

$$0 \longrightarrow M \longrightarrow E \longrightarrow M_1 \longrightarrow 0\,.$$

We have

$$\mu(M) \geq \mu(E) = \frac{g-1}{2} \geq \mu(M_1) \tag{5.1}$$

$$d = c_2(E) = c_1(M).c_1(M_1) + c_2(M) + c_2(M_1) \tag{5.2}$$

Lemma 5.4. *Suppose $C \in |H|_s$ has Clifford index $\gamma = \gamma(C)$. Then if E is as above, we have $d \geq \frac{\gamma}{2} + 4 + \frac{g-1}{2}$.*

Proof. From Proposition 5.1 and Proposition 5.3, we see that $c_1(M).c_1(M_1) \geq \gamma + 2$. Stability of M and M_1 give $-2 \leq \langle \nu(M_{(1)}), \nu(M_{(1)}) \rangle = 4c_2(M_{(1)}) - c_1(M_{(1)})^2 - 8$, thus $c_2(M_{(1)}) \geq \frac{3}{2} + \frac{c_1(M_{(1)})^2}{4}$.

We have

$$\frac{c_1(M)^2 + c_1(M_1)^2}{4} + \frac{c_1(M).c_1(M_1)}{2} = \frac{\mu(M) + \mu(M_1)}{2}$$
$$= \frac{(c_1(M) + c_1(M_1))^2}{4} = \mu(E) = \frac{g-1}{2}.$$

We now calculate

$$\begin{aligned} d &= c_1(M).c_1(M_1) + c_2(M) + c_2(M_1) \\ &\geq c_1(M).c_1(M_1) + 3 + \frac{c_1(M)^2 + c_1(M_1)^2}{4} \\ &= c_1(M).c_1(M_1) + 3 + \frac{g-1}{2} - \frac{c_1(M).c_1(M_1)}{2} \\ &\geq \frac{\gamma+2}{2} + 3 + \frac{g-1}{2}, \end{aligned}$$

as claimed. ∎

5.1.2. Filtration $3 \subset 4$

We assume $E = E_{C,A}$ is unstable with terminal filtration $0 \subset M \subset E$ with M a stable rank 3 torsion free sheaf. Thus E sits in an extension

$$0 \longrightarrow M \longrightarrow E \longrightarrow N \otimes I_\xi \longrightarrow 0$$

where N is a line bundle and I_ξ is the ideal sheaf of a 0-dimensional subscheme $\xi \subset S$ of length $l(\xi) = d - c_1(M).c_1(N)$. We have

$$\mu(M) \geq \mu(E) = \frac{g-1}{2} \geq \mu(N) \tag{5.3}$$

$$c_1(H) = c_1(E) = c_1(M) + c_1(N) \tag{5.4}$$

$$d = c_2(E) = c_1(N).c_1(M) + c_2(M) + l(\xi) \tag{5.5}$$

Lemma 5.5. *Suppose $C \in |H|_s$ has Clifford index $\gamma = \gamma(C)$. Then if E is as above, we have $d \geq \frac{2}{3}(\gamma+2) + \frac{g}{2} + \frac{13}{6}$.*

Proof. From Proposition 5.1 and Proposition 5.3, we see that $c_1(N).c_1(M) \geq \gamma + 2$.

As M is stable, we have $-2 \leq \langle \nu(M), \nu(M) \rangle = 6c_2(M) - 2c_1(M)^2 - 18$, thus $c_2(M) \geq \frac{8 + c_1(M)^2}{3}$. Thus

$$\begin{aligned} d &= c_1(N).c_1(M) + c_2(M) + l(\xi) \\ &\geq c_1(N).c_1(M) + \frac{c_1(M)^2}{3} + \frac{8}{3} \\ &\geq c_1(N).c_1(M) + \frac{g-1}{2} - \frac{c_1(N).c_1(M)}{3} + \frac{8}{3} \\ &\geq \frac{2}{3}(\gamma+2) + \frac{g}{2} + \frac{13}{6}, \end{aligned}$$

as desired. ∎

5.1.3. Filtration $1 \subset 2 \subset 4$

We assume E has a terminal filtration $0 \subset N \subset M \subset E$ with $\mathrm{rk}(N) = 1$, $\mathrm{rk}(M) = 2$, and $E/M = M_1$ a stable torsion free sheaf. Furthermore, we have

$$\mu(N) \geq \mu(M) \geq \mu(E) = \frac{g-1}{2} \geq \mu(M_1) \tag{5.6}$$

$$\mu(M) \geq \mu(M/N) \geq \mu(E/N) \geq \mu(M_1) \tag{5.7}$$

$$d = c_2(E) = c_2(M) + c_2(M_1) + c_1(M).c_1(M_1) \tag{5.8}$$

$$= c_1(N).c_1(M/N) + c_1(N).c_1(M_1) + c_1(M/N).c_1(M_1) + c_2(M_1)$$

Moreover, as M_1 is stable, we have

$$-2 \leq \langle \nu(M_1), \nu(M_1) \rangle = c_1(M_1)^2 - 4\chi(M_1) + 8 = 4c_2(M_1) - c_1(M_1)^2 - 8$$

thus $c_2(M_1) \geq \frac{3}{2} + \frac{c_1(M_1)^2}{4}$. Therefore we have

$$d \geq \frac{3}{2} + \frac{c_1(M_1)^2}{4} + c_1(N).c_1(M/N) + c_1(N).c_1(M_1) + c_1(M/N).c_1(M_1). \tag{5.9}$$

Lemma 5.6. *Suppose E is as above. Then $\det M_1 \otimes \mathcal{O}_C$ contributes to $\gamma(C)$ and one of the following occurs:*

(a) *$N \otimes \mathcal{O}_C$ and $(M/N) \otimes \mathcal{O}_C$ contribute to $\gamma(C)$;*

(b) *$c_1(N).(c_1(M_1) + c_1(M/N)) \geq \frac{g-1}{2} + 2$ and either $(M/N) \otimes \mathcal{O}_C$ contributes to $\gamma(C)$ or*
$c_1(M/N).(c_1(N) + c_1(M_1)) \geq g$;

(c) *$N \otimes \mathcal{O}_C$ contributes to $\gamma(C)$ and $c_1(M/N).(c_1(N) + c_1(M_1)) \geq 2 + \frac{c_1(M).c_1(M_1)}{2} + \frac{c_1(M_1)^2}{2}$;*

(d) *$c_1(N).c_1(M/N) \geq \frac{g+3}{2}$.*

Proof. From Proposition 5.1 and Proposition 5.3, we see that $\det M_1 \otimes \mathcal{O}_C$ contributes to $\gamma(C)$.

We have the following four cases:

(i) $h^0(S, M/N), h^0(S, N) \geq 2$

(ii) $h^0(S, M/N) \geq 2$ and $h^0(S, N) < 2$

(iii) $h^0(S, M/N) < 2$ and $h^0(S, N) \geq 2$

(iv) $h^0(S, M/N), h^0(S, N) < 2$

In case (i), we have $h^0(S, H \otimes (M/N)^\vee) = h^0(S, \det M_1 \otimes N) \geq 2$ and $h^0(S, H \otimes N^\vee) = h^0(S, \det M_1 \otimes M/N) \geq 2$ as $\det M_1$ has global sections. Thus we are in case (a) of the lemma.

In case (ii), we see that $\chi(N) < 2$, hence $c_1(N)^2 \leq -2$, and we calculate

$$
\begin{aligned}
c_1(N).(c_1(M_1).c_1(M/N)) &= c_1(N).(c_1(E) - c_1(N)) \\
&= \mu(N) - c_1(N)^2 \geq \mu(E) + 2 = \frac{g-1}{2} + 2,
\end{aligned}
$$

thus the first statement of case (b) is proved. We now observe that $c_1(N \otimes \det M_1)^2 > c_1(M/N)^2$ which follows from the computation

$$
c_1(N \otimes \det M_1)^2 - c_1(M/N)^2 \geq 2\mu(M_1) > 0.
$$

If $c_1(N \otimes \det M_1)^2 < 0$, then also $c_1(M/N)^2 < 0$, and we calculate

$$
\begin{aligned}
2g - 2 = c_1(E)^2 &= (c_1(N) + c_1(M/N) + c_1(M_1))^2 \\
&= c_1(N \otimes \det M_1)^2 + 2c_1(N \otimes \det M_1).c_1(M/N) + c_1(M/N)^2 \\
&< 2(c_1(N) + c_1(M_1)).c_1(M/N),
\end{aligned}
$$

thus $c_1(M/N).(c_1(N) + c_1(M_1)) \geq g$. Else $c_1(N \otimes \det M_1)^2 \geq 0$ and so $h^0(S, H \otimes (M/N)^\vee) = h^0(S, N \otimes \det M_1) \geq 2$ and so M/N contributes to $\gamma(C)$. Thus we are in case (b).

In case (iii), since $\det E/N \cong \det M_1 \otimes M/N$, we have $h^0(S, \det M_1 \otimes M/N) \geq 2$. Thus as $h^0(S, N) \geq 2$, we see that $N \otimes \mathcal{O}_C$ contributes to $\gamma(C)$. Therefore, as $h^0(S, M/N) < 2$, we have $c_1(M/N)^2 \leq -2$.

In cases (iii) and (iv), we have $c_1(M/N)^2 \leq -2$. We now calculate

$$\begin{aligned}
2g - 2 &= c_1(E)^2 \\
&= c_1(M/N)^2 + c_1(N)^2 + c_1(M_1)^2 \\
&\quad + 2c_1(M/N).c_1(N) + 2c_1(M/N).c_1(M_1) + 2c_1(N).c_1(M_1) \\
&\leq c_1(N)^2 + c_1(M_1)^2 \\
&\quad + 2c_1(M/N).c_1(N) + 2c_1(M/N).c_1(M_1) + 2c_1(N).c_1(M_1) - 2 \\
&\leq c_1(N)^2 + g - 3 + 2c_1(M/N).c_1(N),
\end{aligned}$$

thus

$$c_1(N).c_1(M/N) \geq \frac{g+1}{2} - \frac{c_1(N)^2}{2}. \tag{5.10}$$

In case (iii), we observe that since

$$c_1(M/N).(c_1(N) + c_1(M_1)) + c_1(M/N)^2 = \mu(M/N) \geq \mu(E/N) = \frac{(c_1(E/N)).(c_1(E))}{3},$$

we have

$$\begin{aligned} c_1(M/N).(c_1(N)+c_1(M_1)) \geq & -c_1(M/N)^2 + \frac{c_1(M_1)^2}{3} \\ & + \frac{c_1(M/N)^2}{3} + \frac{c_1(M).c_1(M_1)}{3} \\ & + \frac{c_1(M/N).(c_1(N)+c_1(M_1))}{3}. \end{aligned}$$

And subtracting $c_1(M/N).(c_1(N)+c_1(M_1))/3$ from both sides and multiplying by $3/2$ yields

$$c_1(M/N).(c_1(N)+c_1(M_1)) \geq -c_1(M/N)^2 + \frac{c_1(M).c_1(M_1)}{2} + \frac{c_1(M_1)^2}{2}.$$

Noting that $c_1(M/N)^2 \leq -2$ shows we are in case (c).

In case (iv), as $h^0(S,N), h^0(S,M/N) < 2$, we have $c_1(N)^2, c_1(M/N)^2 \leq -2$, thus Equation (5.10) gives $c_1(N).c_1(M/N) \geq \frac{g+1}{2} - \frac{c_1(N)^2}{2} \geq \frac{g+1}{2} + 1 = \frac{g+3}{2}$, and we are in case (d). □

Lemma 5.7. *With E as above, if general curves in $|H|_s$ have Clifford index $\gamma = \gamma(C)$, and $m = D^2$ is the minimum self-intersection of an effective classes $D \in \mathrm{Pic}(S)$ (i.e. there are no curves of genus $g' < \frac{m+2}{2}$ on S), then we have $d \geq \frac{5}{4}\gamma + \frac{m}{2} + 5$ or $d \geq 5 + \frac{3}{2}\gamma$. Moreover, when A is primitive, then we can assume $m \geq 2$.*

Proof. We write

$$2d \geq 3 + \frac{c_1(M_1)^2}{2} + c_1(N).c_1(E/N) + c_1(M/N).(c_1(N)+c_1(M_1)) + c_1(M).c_1(M_1),$$

and apply bounds to each of the terms. From Proposition 5.1, we see that $c_1(N).c_1(E/N) \geq \gamma + 2$, and $c_1(M).c_1(M_1) \geq \gamma + 2$. In cases (a), (b), we have $c_1(M/N).(c_1(N)+c_1(M_1)) \geq \gamma + 2$. In case (c), we have $d \geq \frac{5}{4}\gamma + \frac{m}{2} + 5$. Finally, in case (d), we have $d \geq 2 + c_1(N).c_1(M/N) + c_1(M).c_1(M_1) \geq 2 + \frac{g+13}{2} + \gamma + 2$. And in any case, we have the desired inequality.

When A is primitive, M_1 is a gLM of type (II), and as $c_1(M_1)^2 \geq 0$ we have $c_2(M_1) > 0$, thus we cannot have $c_1(M_1)^2 = 0$ by Remark 3.26. Therefore m can be taken to be at least 2. $\square$

5.1.4. Filtration $1 \subset 3 \subset 4$

We assume E has a terminal filtration $0 \subset N \subset M \subset E$ with $\mathrm{rk}(N) = 1$, $\mathrm{rk}(M) = 3$, and M/N a stable torsion free sheaf, and we call $E/M = N_1$. Furthermore, we have

$$\mu(N) \geq \mu(M) \geq \mu(E) \geq \mu(E/N) \geq \mu(N_1) \tag{5.11}$$

$$\mu(M) \geq \mu(M/N) \geq \mu(E/N) \tag{5.12}$$

$$d = c_2(E) = c_2(M/N) + c_1(M/N).c_1(N) + c_1(N).c_1(N_1) + c_1(N_1).c_1(M/N) \tag{5.13}$$

Moreover, since M/N is stable, we have

$$-2 \leq \langle \nu(M/N), \nu(M/N) \rangle = c_1(M/N)^2 - 4\chi(M/N) + 8 = 4c_2(M/N) - c_1(M/N)^2 - 8$$

thus $c_2(M/N) \geq \frac{3}{2} + \frac{c_1(M/N)^2}{4}$.

Lemma 5.8. *Suppose E is as above. Then $N_1 \otimes \mathcal{O}_C$ contributes to $\gamma(C)$, and one of the following occurs:*

(a) *$N \otimes \mathcal{O}_C$ and $\det(M/N) \otimes \mathcal{O}_C$ contribute to $\gamma(C)$;*

(b) *$c_1(N).(c_1(N_1) + c_1(M/N)) \geq \frac{g+3}{2} \geq \gamma(C) + 2$ and either $\det(M/N) \otimes \mathcal{O}_C$ contributes to $\gamma(C)$ or $\frac{c_1(M/N)^2}{2} + c_1(M/N).(c_1(N) + c_1(N_1)) \geq g$;*

(c) *$N \otimes \mathcal{O}_C$ contributes to $\gamma(C)$ and $\frac{c_1(M/N)^2}{2} + c_1(M/N).c_1(N) \geq \frac{1}{2}c_1(N).(c_1(N_1) + c_1(M/N))$;*

(d) $\frac{c_1(M/N)^2}{2} + c_1(M/N).c_1(N) \geq g+1$.

Proof. From Proposition 5.1 and Proposition 5.3, we see that $N_1 \otimes \mathcal{O}_C$ contributes to $\gamma(C)$ and $h^2(S, \det M/N) = h^2(S, M/N) = h^2(S, N) = 0$.

We have the following four cases:

(i) $h^0(S, \det M/N), h^0(S, N) \geq 2$

(ii) $h^0(S, \det M/N) \geq 2$ and $h^0(S, N) < 2$

(iii) $h^0(S, \det M/N) < 2$ and $h^0(S, N) \geq 2$

(iv) $h^0(S, \det M/N), h^0(S, N) < 2$

In case (i), we have $h^0(S, H \otimes N^\vee) = h^0(S, \det M/N \otimes N_1) \geq 2$, and $h^0(S, H \otimes \det M/N^\vee) = h^0(S, N \otimes N_1) \geq 2$ as $\det M/N$, N, and N_1 have global sections. Thus we are in case (a) of the lemma.

In case (ii), we see that $\chi(N) < 2$, thus $c_1(N) \leq -2$, and we calculate

$$\begin{aligned} c_1(N).(c_1(N_1) + c_1(M/N)) =& c_1(N).(c_1(E) - c_1(N)) \\ &= \mu(N) - c_1(N)^2 \geq \mu(E) + 2 = \frac{g+3}{2}. \end{aligned}$$

- If $c_1(N \otimes N_1)^2, c_1(M/N)^2 \geq 0$, then $\det M/N$ contributes to $\gamma(C)$ as $h^0(S, N \otimes N_1) = h^0(S, H - \det M/N) \geq 2$.
- If $c_1(N \otimes N_1)^2 \geq 2$ and $c_1(M/N)^2 < 0$, then as above $\det M/N$ contributes to $\gamma(C)$.
- If $c_1(N \otimes N_1)^2 < 0$ and $c_1(M/N)^2 \geq 0$ then we cannot say if $\det M/N$ contributes

to $\gamma(C)$ as above. However, we calculate

$$\begin{aligned}2g-2=c_1(E)^2=&(c_1(M/N)+c_1(N\otimes N_1))^2\\&=c_1(M/N)^2+2c_1(M/N).(c_1(N)+c_1(N_1))+c_1(N\otimes N_1)^2\\&<c_1(M/N)^2+2c_1(M/N).(c_1(N)+c_1(N_1)),\end{aligned}$$

thus $\frac{c_1(M/N)^2}{2}+c_1(M/N).(c_1(N)+c_1(N_1))\geq g$.

- If $c_1(N\otimes N_1)^2, c_1(M/N)^2<0$, then the same calculation as above yields $\frac{c_1(M/N)^2}{2}+c_1(M/N).(c_1(N)+c_1(N_1))\geq g$.

Thus we are in case (b) of the lemma.

In case (iii), since $\det E/N=N_1\otimes\det M/N$, [56, Lemma 3.3] implies that $h^0(S,N_1\otimes\det M/N)\geq 2$. Thus since $h^0(S,N)\geq 2$, we see that $N\otimes\mathcal{O}_C$ contributes to $\gamma(C)$. Furthermore, as $c_1(M/N)^2+c_1(M/N).c_1(N)\geq c_1(N_1)^2+c_1(N_1).c_1(N)$ and $c_1(N_1)^2\geq 0>c_1(M/N)^2$, we have $c_1(M/N)^2+c_1(M/N).c_1(N)\geq c_1(N_1).c_1(N)$. Thus

$$\begin{aligned}c_1(M/N)^2&+c_1(M/N).c_1(N)-\frac{1}{2}(c_1(N).(c_1(N_1)+c_1(M/N)))\\&\geq c_1(M/N)^2+\frac{c_1(M/N).c_1(N)}{2}-\frac{c_1(N).c_1(N_1)}{2}\\&\geq\frac{c_1(M/N)^2}{2},\end{aligned}$$

thus

$$\frac{c_1(M/N)^2}{2}+c_1(M/N).c_1(N)\geq\frac{1}{2}c_1(N).(c_1(N_1)+c_1(M/N)),$$

and we are in case (c).

In case (iv), we see that $c_1(N)^2, c_1(M/N)^2 \leq -2$. We calculate

$$\begin{aligned}2g-2 = c_1(E)^2 &= (c_1(N)+c_1(N_1)+c_1(M/N))^2 \\ &\leq c_1(N_1)^2 + c_1(M/N)^2 \\ &\quad + 2c_1(N).c_1(N_1) + 2c_1(N).c_1(M/N) + 2c_1(N_1).c_1(M/N) - 2 \\ &\leq g-1+2c_1(N).c_1(M/N) + c_1(M/N)^2 - 2,\end{aligned}$$

thus $\frac{c_1(M/N)^2}{2} + c_1(N).c_1(M/N) \geq g+1$, and we are in case (d). $\square$

Remark 5.9. From the second half of the proof of Proposition 5.1, we see that in the situation above, if $C \in |H|_s$ has Clifford index $\gamma = \gamma(C)$, and if $\det M/N$ contributes to $\gamma(C)$, then we have $c_1(M/N).(c_1(N)+c_1(N_1)) \geq \gamma + 2 + 2h^1(S, \det M/N)$.

Lemma 5.10. *With E as above, if general curves in $|H|_s$ have Clifford index $\gamma = \gamma(C)$, we have $d \geq \frac{3}{2}\gamma + 5$.*

Proof. We first see that if $c_1(M/N)^2 \geq 0$, then we are in cases (a) or (b) of the above lemma. Furthermore, we have $c_2(M/N) \geq 2$. Thus in case (a), we have

$$\begin{aligned}2d &\geq 2(c_2(M/N) + c_1(M/N).c_1(N) + c_1(N).c_1(N_1) + c_1(N_1).c_1(M/N)) \\ &\geq 4 + 2c_1(M/N).c_1(N) + 2c_1(N).c_1(N_1) + 2c_1(N_1).c_1(M/N) \\ &= 4 + c_1(M/N).(c_1(N)+c_1(N_1)) + c_1(N).(c_1(N_1)+c_1(M/N)) \\ &\quad + c_1(N_1).(c_1(M/N)+c_1(N)) \\ &\geq 4 + 3(\gamma+2),\end{aligned}$$

where the last inequality comes from Proposition 5.1. Thus $d \geq \frac{3}{2}\gamma + 5$. In case (b), we calculate as in case (a) and get $d \geq \frac{3}{2}\gamma + 5$ or

$$\begin{aligned} 2d &\geq 2\left(c_1(N).c_1(N_1) + c_1(N).c_1(M/N) + c_1(N_1).c_1(M/N) + \frac{c_1(M/N)^2}{4}\right) \\ &\geq g + c_1(N).c_1(M/N) + 2c_1(N).c_1(N_1) + c_1(N_1).c_1(M/N) \\ &\geq g + 2(\gamma + 2), \end{aligned}$$

hence $d \geq \gamma + 2 + \frac{g}{2} > \frac{3}{2}\gamma + 5$.

If $c_1(M/N)^2 < 0$, in case (d), we have

$$\begin{aligned} d &\geq \frac{3}{2} + \frac{g+1}{2} + \frac{c_1(N).c_1(M/N)}{2} + c_1(M/N).c_1(N_1) + c_1(N).c_1(N_1) \\ &\geq \frac{g+4}{2} + k + \frac{g+1}{2} - \frac{c_1(M/N)^2}{4} \\ &\geq \gamma + 2 + g + \frac{7}{2}. \end{aligned}$$

If $0 > c_1(M/N)^2 \geq -6$, then $c_2(M/N) \geq 0$, thus $\chi(\det M/N) \leq 1$. Therefore $h^1(S, \det M/N) + 1 \geq h^0(S, \det M/N)$. Calculating as above, we see that

- in case (a), we have $d \geq \frac{3}{2}\gamma + 5$;
- in case (b), we have $d \geq \frac{3}{2}\gamma + 5$ or $d \geq \gamma + \frac{7}{2} + \frac{g+2}{2}$; and,
- in case (c), we have $d \geq \frac{3}{2}\gamma + 5$.

If $c_2(M/N) < 0$, then the stability of M/N implies that $c_1(M/N)^2 \leq -8$ and

$$-2 \leq \langle \nu(M/N), \nu(M/N) \rangle = c_1(M/N)^2 + 8 - 4\chi(M/N) \leq -4\chi(M/N),$$

whereby $\chi(M/N) \leq 0$. We now consider inequalities associated with various filtrations that lead to the terminal $1 \subset 3 \subset 4$ filtration of E.

If the JH filtration of E is $1 \subset 3 \subset 4$, then we have $p(E) = p(M/N)$, which gives an equality of normalized Euler characteristics

$$\frac{\chi(M/N)}{2} = \frac{\chi(E)}{4} = \frac{g-\gamma+1}{4}.$$

Thus $0 \geq 2\chi(M/N) = g - d + 7$, and hence $d \geq g + 7$.

If the HN filtration of E is $0 \subset M \subset E$ with $\mathrm{rk}(M) = 3$ and M properly semistable, then the JH filtration of M is $0 \subset N \subset M$. Hence $\mu(M/N) = \mu(M)$ and $\mu(M) > \mu(E)$. Thus

$$\frac{c_1(M/N)^2}{2} + \frac{c_1(M/N).c_1(N \otimes N_1)}{2} = \mu(M/N) > \mu(E) = \frac{g-1}{2},$$

hence

$$\begin{aligned} d &\geq \frac{3}{2} + \frac{c_1(M/N)^2}{4} + c_1(M/N).c_1(N \otimes N_1) + c_1(N).c_1(N_1) \\ &\geq \frac{3}{2} + \frac{g-1}{2} - \frac{c_1(M/N)^2}{4} + \frac{c_1(M/N).(c_1(N) + c_1(N_1))}{2} \\ &\geq \frac{3}{2} + \frac{g-1}{2} + \frac{c_1(N).(c_1(N_1) + c_1(M/N))}{2} + \frac{c_1(N_1).(c_1(N) + c_1(M/N))}{2} \\ &\geq \frac{3}{2} + \frac{g-1}{2} + \gamma + 1 \end{aligned}$$

where the last inequality comes from the fact that N_1 contributes to $\gamma(C)$, and that in cases (a),(b), and (c) $c_1(N).(c_1(N_1) + c_1(M/N)) \geq \gamma + 2$.

If the HN filtration of E is $0 \subset N \subset E$ with E/N properly semistable and the JH filtration of E/N is $0 \subset \overline{M} \subset E/N$ with $\mathrm{rk}(\overline{M}) = 2$, then we have an equality of normalized Euler characteristics

$$\frac{\chi(E) - \chi(N)}{3} = \frac{\chi(E/N)}{3} = \frac{\chi(\overline{M})}{2} = \frac{\chi(M/N)}{2}.$$

Thus $\chi(E) = g - \gamma + 1 = \frac{3\chi(M/N)}{2} + \chi(N)$, where $\gamma = d - 6$ is the Clifford index of the g^3_d on C. From the short exact sequence

$$0 \longrightarrow N \longrightarrow E \longrightarrow E/N \longrightarrow 0\,,$$

we have $\chi(N) = h^0(S,E) - h^0(S,E/N) \leq g-\gamma-1$ as $h^0(S,E/N) \geq 2$. Therefore

$$g-\gamma+1 = \chi(E) \leq \frac{3\chi(M/N)}{2} + g - \gamma - 1,$$

and thus $2 \leq \frac{3}{2}\chi(M/N) \leq 0$, which is a contradiction. Thus this does not occur, and in all cases we have at least $d \geq \frac{3}{2}\gamma + 5$, as claimed. □

5.1.5. Filtration $2 \subset 3 \subset 4$

We assume E has a terminal filtration $0 \subset N \subset M \subset E$ with N a stable torsion free sheaf of rank $\mathrm{rk}(N) = 2$, $\mathrm{rk}(M) = 3$, and $N_1 = E/M$ a line bundle. Furthermore, we have

$$\mu(N) \geq \mu(M) \geq \mu(E) = \frac{g-1}{2} \geq \mu(N_1) \tag{5.14}$$

$$\mu(M) \geq \mu(M/N) \geq \mu(E/N) \geq \mu(N_1) \tag{5.15}$$

$$d = c_2(E) = c_2(N) + c_1(N).c_1(M/N) + c_1(N).c_1(N_1) + c_1(M/N).c_1(N_1) \tag{5.16}$$

Moreover, as N is stable, we have $c_2(N) \geq \frac{3}{2} + \frac{c_1(N)^2}{4}$.

Lemma 5.11. *Suppose E is as above. Then $N_1 \otimes \mathcal{O}_C$ contributes to $\gamma(C)$ and one of the following occurs:*

(a) *$(\det N) \otimes \mathcal{O}_C$ and $(M/N) \otimes \mathcal{O}_C$ contribute to $\gamma(C)$;*

(b) *$c_1(N).(c_1(N_1) + c_1(M/N)) \geq g+1$ and either $(M/N) \otimes \mathcal{O}_C$ contributes to $\gamma(C)$ or $c_1(M/N).(c_1(N) + c_1(N_1)) \geq g$;*

(c) $(\det N) \otimes \mathcal{O}_C$ *contributes to* $\gamma(C)$*, we can assume* $c_1(N)^2 \geq 0$ *and* $c_1(M/N).c_1(N) \geq \frac{1}{2}c_1(N).(c_1(M/N) + c_1(N_1))$*;*

(d) $c_1(N)^2 \leq -2$ *and* $\frac{c_1(N)^2}{2} + c_1(M/N).c_1(N) \geq \frac{g+1}{2}$.

Proof. From Proposition 5.1 and Proposition 5.3, we see that $N_1 \otimes \mathcal{O}_C$ contributes to $\gamma(C)$ and $h^2(S, \det N) = h^2(S, \det M) = h^2(S, M/N) = h^2(S, \det E/N) = 0$.

We have the following four cases:

(i) $h^0(S, M/N), h^0(S, \det N) \geq 2$

(ii) $h^0(S, M/N) \geq 2$ and $h^0(S, \det N) < 2$

(iii) $h^0(S, M/N) < 2$ and $h^0(S, \det N) \geq 2$

(iv) $h^0(S, M/N), h^0(S, \det N) < 2$.

In case (i), as N_1 has global sections, and $H - c_1(M/N) = c_1(N) + c_1(N_1)$ and $H - c_1(N) = c_1(N_1) + c_1(M/N)$, we see that both $(\det N) \otimes \mathcal{O}_C$ and $(M/N) \otimes \mathcal{O}_C$ contribute to $\gamma(C)$, and we are in case (a).

In case (ii), we have $\chi(N) < 2$, hence $c_1(N)^2 \leq -2$, and we calculate

$$\begin{aligned} c_1(N).(c_1(N_1) + c_1(M/N)) &= c_1(N).(c_1(E) - c_1(N)) \\ &= 2\mu(N) - c_1(N)^2 \geq g - 1 + 2 = g + 1 \end{aligned}$$

We now observe that $c_1(\det N \otimes N_1)^2 \geq c_1(M/N)^2$ which follows from the following calculation

$$\begin{aligned} c_1(\det N \otimes N_1)^2 - c_1(M/N)^2 &= c_1(N)^2 + 2c_1(N).c_1(N_1) + c_1(N_1)^2 - c_1(M/N)^2 \\ &= 2\mu(N) + \mu(N_1) - \mu(M/N) \geq \mu(N) + \mu(N_1) > 0. \end{aligned}$$

If $c_1(\det N \otimes N_1)^2 < 0$, then also $c_1(M/N) < 0$, and we calculate

$$
\begin{aligned}
2g-2 = c_1(E)^2 &= (c_1(N) + c_1(M/N) + c_1(N_1))^2 \\
&= c_1(\det N \otimes N_1)^2 + 2c_1(\det N \otimes N_1).c_1(M/N) + c_1(M/N)^2 \\
&< 2(c_1(N) + c_1(N_1)).c_1(M/N),
\end{aligned}
$$

thus $c_1(M/N).(c_1(N) + c_1(N_1)) \geq g$. Else $c_1(\det N \otimes N_1)^2 \geq 0$, and so $h^0(S, H \otimes (M/N)^\vee) = h^0(S, \det N \otimes N_1) \geq 2$, whereby $M/N \otimes \mathcal{O}_C$ contributes to $\gamma(C)$. Thus we are in case (b).

In case (iii), since $\det E/N \cong \det M/N \otimes N_1$, we have $h^0(S, \det M/N \otimes N_1) \geq 2$ by [56, Lemma 3.3]. Thus as $h^0(S, \det N) \geq 2$, we have that $\det N \otimes \mathcal{O}_C$ contributes to $\gamma(C)$. Thus proving the first statement of case (c).

In cases (iii) and (iv), as $h^0(S, M/N) < 2$ we have $c_1(M/N)^2 \leq -2$. We now calculate

$$
\begin{aligned}
2g-2 = c_1(E)^2 &= (c_1(N) + c_1(M/N) + c_1(N_1))^2 \\
&= c_1(N)^2 + c_1(M/N)^2 + c_1(N_1)^2 \\
&\quad + 2c_1(N).c_1(M/N) + 2c_1(M/N).c_1(N_1) + 2c_1(N).c_1(N_1) \\
&\leq c_1(N)^2 + c_1(N_1)^2 \\
&\quad + 2c_1(N).c_1(M/N) + 2c_1(M/N).c_1(N_1) + 2c_1(N).c_1(N_1) - 2 \\
&\leq g - 1 + c_1(N)^2 + 2c_1(M/N).c_1(N) - 2,
\end{aligned}
$$

thus $\frac{c_1(N)^2}{2} + c_1(M/N).c_1(N) \geq \frac{g+1}{2}$. If $c_1(N)^2 \leq -2$, we are in case (d).

From now on, we assume $c_1(N)^2 \geq 0$. From the inequality $\mu(M/N) \geq \mu(N_1)$, we see that

$$\begin{aligned}
c_1(N).c_1(M/N) &> c_1(M/N)^2 + c_1(N).c_1(M/N) \\
&\geq c_1(N_1)^2 + c_1(N_1).c_1(M/N) \\
&\geq c_1(N_1).c_1(M/N).
\end{aligned}$$

Thus

$$\begin{aligned}
&c_1(M/N).c_1(N) - \frac{1}{2}c_1(N).(c_1(M/N) + c_1(N_1)) \\
&= \frac{1}{2}(c_1(N).c_1(M/N) - c_1(N).c_1(N_1)) > 0,
\end{aligned}$$

and we are in case (c).

□

Lemma 5.12. *With E as above, if general curves in $|H|_s$ have Clifford index $\gamma = \gamma(C)$, we have $d \geq 5 + \frac{3}{2}\gamma$.*

Proof. The proof follows the same argument as Lemma 5.7. □

5.1.6. Filtration $1 \subset 2 \subset 3 \subset 4$

We suppose E has a terminal filtration of the form

$$0 = E_0 \subset E_1 \subset E_2 \subset E_3 \subset E_4 = E,$$

where $\mathrm{rk}(E_i) = i$, and E_i/E_{i+1} are torsion free sheaves of rank 1. Furthermore, we have

$$\mu(E_1) \geq \mu(E_2) \geq \mu(E_3) \geq \mu(E) = \frac{g-1}{2} \geq \mu(E/E_3) \tag{5.17}$$

$$\mu(E_1) \geq \mu(E_2/E_1) \geq \mu(E_3/E_2) \geq \mu(E/E_3) \tag{5.18}$$

$$\mu(E_i/E_j) \geq \mu(E/E_3) \text{ for } 1 \leq j < i \leq 4 \tag{5.19}$$

$$\begin{aligned} d = {} & c_1(E/E_3).(c_1(E_1) + c_1(E_2/E_1) + c_1(E_3/E_2)) + c_1(E_1).c_1(E_3/E_2) \\ & + c_1(E_2/E_1).c_1(E_3/E_2) + c_1(E_1).c_1(E_2/E_1) \end{aligned} \tag{5.20}$$

Letting $e_i := c_1(E_i/(E_{i-1}))$, be the Chern roots of E, and writing

$$e_i + e_j := c_1(E_i/E_{i-1} \otimes E_j/E_{j-1}),$$

we have

$$\begin{aligned} 4d = {} & e_1(e_2 + e_3 + e_4) + (e_1 + e_2).(e_3 + e_4) + (e_1 + e_2 + e_3).e_4 \\ & + (e_1 + e_4).(e_2 + e_3) + (e_1 + e_3).(e_2 + e_4) + (e_1 + e_3 + e_4).e_2 + (e_1 + e_2 + e_4).e_3 \end{aligned}$$

Lemma 5.13. *With E as above, if general curves in $|H|_s$ have Clifford index $\gamma = \gamma(C)$,*

$$m := \min\{D^2 | D \in \mathrm{Pic}(S), D^2 \geq 0,\ D \text{ is effective}\}$$

(i.e. there are no curves of genus $g' < \frac{m+2}{2}$ on S), and

$$\mu = \min\{\mu(D) | D \in \mathrm{Pic}(S), D^2 \geq 0, \mu(D) > 0\},$$

we have $d \geq \frac{5}{4}\gamma + \frac{\mu}{2} + \frac{m}{2} + \frac{9}{2}$.

Proof. From Proposition 5.1 and Proposition 5.3, we see that $\det(E/E_i) \otimes \mathcal{O}_C$ contributes to $\gamma(C)$, and so we have $e_1(e_2+e_3+e_4) \geq \gamma+2$, $(e_1+e_2).(e_3+e_4) \geq \gamma+2$, and $(e_1+e_2+e_3).e_4 \geq \gamma+2$. We also have $h^2(S,F)=0$ for $F=\det(E_i/E_j)$ and $F = E/E_3, \det E_i$.

It remains to bound the other four terms.

To bound $(e_2+e_3).(e_1+e_4)$, we note that $\mu(e_2+e_3) \geq \mu+\mu(e_3) \geq \mu+\mu(E/E_2)$, and thus

$$\begin{aligned}
&(e_2+e_3)^2+(e_1+e_4).(e_2+e_3)\\
&\geq \mu + \frac{(e_1+e_2).(e_3+e_4)}{2} + \frac{(e_3+e_4)^2}{2}\\
&\geq \mu + \frac{\gamma+2}{2} + \frac{(e_3+e_4)^2}{2}.
\end{aligned}$$

Furthermore, we note that $\mu(e_1+e_4) = \mu(e_1)+\mu(e_4) \geq \frac{g-1}{2}+\mu$, whereby

$$(e_1+e_4)^2+(e_1+e_4).(e_2+e_3) \geq \gamma.$$

Now if $h^0(S,e_1+e_4) < 2$ then by considering the Euler characteristic we have $(e_1+e_4)^2 \leq -2$, and thus $(e_1+e_4).(e_2+e_3) \geq \gamma+2$. If $h^0(S,e_2+e_3) < 2$ then $(e_2+e_3)^2 \leq -2$, and we have $(e_1+e_4).(e_2+e_3) \geq 2+\mu+\frac{\gamma+2}{2}+\frac{(e_3+e_4)^2}{2}$. By assumption, $(e_3+e_4)^2 \geq m$, hence $(e_1+e_4).(e_2+e_3) \geq 3+\mu+\frac{\gamma}{2}+\frac{m}{2}$ as well. Finally, if $h^0(S,e_1+e_4), h^0(S,e_2+e_3) \geq 2$, and thus they contribute to the $\gamma(C)$, and hence by Proposition 5.1 $(e_1+e_4).(e_2+e_3) \geq \gamma+2$. Therefore in either case, we have $(e_1+e_4).(e_2+e_3) \geq 3+\mu+\frac{\gamma}{2}+\frac{m}{2}$.

To bound $(e_1+e_3).(e_2+e_4)$, we note that $\mu(e_1+e_3) \geq \frac{g-1}{2}$, and hence

$$(e_1+e_3)^2+(e_1+e_3).(e_2+e_4) \geq \frac{g-1}{2}.$$

We also note that $\mu(e_2 + e_4) \geq \mu + \mu(E/E_1) \geq \mu + \mu(E/E_2)$, whereby

$$\begin{aligned}
&(e_2+e_4)^2 + (e_1+e_3).(e_2+e_4) \\
&\geq 1 + \frac{(e_1+e_2).(e_3+e_4)}{2} + \frac{(e_3+e_4)^2}{2} \\
&\geq 1 + \frac{\gamma+2}{2} + \frac{(e_3+e_4)^2}{2}.
\end{aligned}$$

As above, we have $(e_1 + e_3).(e_2 + e_4) \geq 3 + \mu + \frac{\gamma}{2} + \frac{m}{2}$.

To bound $(e_1 + e_3 + e_4).e_2$, we note that $\mu(e_1 + e_3 + e_4) \geq \mu(e_1) \geq \frac{g-1}{2}$ and $\mu(e_2) \geq \mu(E/E_1) \geq \mu(E/E_2)$. Following the same argument as above, we have $(e_1 + e_3 + e_4).e_2 \geq 3 + \frac{\gamma}{2} + \frac{m}{2}$.

To bound $(e_1+e_2+e_4).e_3$, we note that $\mu(e_1+e_2+e_4) \geq \mu(e_1) \geq \frac{g-1}{2}$ and $\mu(e_3) \geq \mu(E/E_2)$. Following the same argument as above, we have $(e_1+e_2+e_4).e_3 \geq 3+\frac{\gamma}{2}+\frac{m}{2}$.

Finally, we have that three of the terms in the expression for $4d$ are bounded below by $\gamma + 2$, two by $3 + \frac{\gamma}{2} + \frac{m}{2}$, and two by $3 + \mu + \frac{\gamma}{2} + \frac{m}{2}$. Thus $d \geq \frac{5}{4}\gamma + \frac{\mu}{2} + \frac{m}{2} + \frac{9}{2}$, as desired. □

Remark 5.14. We note that in the proof above, μ is always at least the minimum slope of the determinant of a quotient of E.

Section 5.2

Lifting g^3_ds

As above, (S, H) is a polarized K3 surface of genus g, $C \in |H|$ is a smooth irreducible curve of general Clifford index $\gamma = \lfloor \frac{g-1}{2} \rfloor$, A is a complete basepoint free g^3_d with $\rho(A) < 0$, and $E = E_{C,A}$ the unstable LM bundle. Having attained the needed bounds on $c_2(E)$, we can prove our lifting results.

Theorem 5.15. *Let (S, H) be a polarized K3 surface of genus $g \neq 2, 3, 4, 8$ and*

$C \in |H|$ a smooth irreducible curve of Clifford index γ. Let

$$m := \min\{D^2 \mid D \in \mathrm{Pic}(S),\ D^2 \geq 0,\ D \text{ is effective}\}$$

(i.e. there are no curves of genus $g' < \frac{m+2}{2}$ on S), and

$$\mu = \min\{\mu(D) \mid D \in \mathrm{Pic}(S),\ D^2 \geq 0,\ \mu(D) > 0\}.$$

If

$$d < \min\left\{\frac{5}{4}\gamma + \frac{\mu}{2} + \frac{m}{2} + \frac{9}{2},\ \frac{5}{4}\gamma + \frac{m}{2} + 5,\ \frac{3}{2}\gamma + 5,\ \frac{\gamma}{2} + \frac{g-1}{2} + 4\right\},$$

then there is a line bundle $M \in \mathrm{Pic}(S)$ adapted to $|H|$ such that $|A| \subseteq |M \otimes \mathcal{O}_C|$ and $\gamma(M \otimes \mathcal{O}_C) \leq \gamma(A)$. Moreover, one has $c_1(M).C \leq \frac{3g-3}{2}$.

Proof. The LM bundle E has $c_2(E) = d$. If $g \neq 2, 3, 4, 8$, then $d < \frac{5g+19}{6}$. By the assumptions on d, the only terminal filtration of E is of type $1 \subset 4$. Thus by Proposition 4.17, the result follows. □

Considering the bounds obtained in Section 5.1, we have also proved the following proposition.

Proposition 5.16. *With A as above, the bundle $E_{C,A}$ only admits a terminal filtration of type $1 \subset 4$, $1 \subset 2 \subset 4$, or $1 \subset 2 \subset 3 \subset 4$.*

Proof. We simply solve $\rho(g, 3, d) < 0$ for d and compare it to the bounds obtained for each terminal filtration. □

Chapter 6

Maximal Brill–Noether Loci via K3 surfaces

In this chapter, we outline a conjecture regarding containments of Brill–Noether loci and verify it in genus ≤ 19, 22, and 23. The sections are taken from [6].

Section 6.1

Maximal Brill–Noether locus conjecture

A question of interest is to determine the stratification of $\mathcal{M}_g$ by Brill–Noether loci and, in particular, to identify those loci that are maximal with respect to containment. For Brill–Noether divisors, this is equivalent to having distinct support, a property that is crucially used by Eisenbud and Harris [23], and further developed by Farkas [26], to give lower bounds on the Kodaira dimension of $\mathcal{M}_{23}$.

There are various trivial containments among the Brill–Noether loci, e.g., $\mathcal{M}^1_{g,2} \subseteq \mathcal{M}^1_{g,3} \subseteq \cdots \subseteq \mathcal{M}^1_{g,k} = \mathcal{M}_g$, where $k \geq \lfloor \frac{g+3}{2} \rfloor$ is at least the generic gonality of a curve of genus g. Likewise, we have $\mathcal{M}^r_{g,d} \subseteq \mathcal{M}^r_{g,d+1}$ by adding a base point to a g^r_d on C. Similarly, by subtracting a point not in the base locus, $\mathcal{M}^r_{g,d} \subseteq \mathcal{M}^{r-1}_{g,d-1}$ when

$\rho(g, r-1, d-1) < 0$, see [27, 55].

Modulo these trivial containments, the *expected maximal Brill–Noether loci* are the $\mathcal{M}^r_{g,d}$, where for fixed r, with $2r \leq d \leq g-1$, d is maximal such that $\rho(g,r,d) < 0$ and $\rho(g, r-1, d-1) \geq 0$. Hence, every Brill–Noether locus is contained in an expected maximal one, and we conjecture that the expected maximal loci are distinct.

Conjecture 6.1 ([6])**.** *In every genus $g \geq 3$, the maximal Brill–Noether loci are the expected ones, except when $g = 7, 8, 9$.*

The conjecture states that at least one component of each expected maximal Brill–Noether locus is not contained in any other Brill–Noether locus, hence the expected maximal loci are indeed the maximal elements in the containment lattice of all Brill–Noether loci. Concretely, this means that given any two expected maximal Brill–Noether loci $\mathcal{M}^r_{g,d}$ and $\mathcal{M}^{r'}_{g,d'}$, there exists a genus g curve admitting a g^r_d but not a $g^{r'}_{d'}$.

In each genus $g = 7, 8, 9$, there is an unexpected containment between the two expected maximal Brill–Noether loci. In genus 8, Mukai [63, Lemma 3.8] proved the unexpected containment $\mathcal{M}^1_{8,4} \subset \mathcal{M}^2_{8,7}$, see Proposition 6.24. In genus 7 and 9, Hannah Larson pointed out the unexpected containments $\mathcal{M}^2_{7,6} \subset \mathcal{M}^1_{7,4}$ and $\mathcal{M}^2_{9,7} \subset \mathcal{M}^1_{9,5}$, see Proposition 6.23 and Proposition 6.25.

Recently, there have been several breakthroughs in the study of Brill–Noether special curves of fixed gonality [16, 27, 43, 51, 52, 69, 70], from which one can deduce that the expected maximal $\mathcal{M}^1_{g,\lfloor \frac{g+1}{2} \rfloor}$ is not contained in any of the other expected maximal loci and hence is maximal, see Section 6.2. Additionally, following the work of Farkas [27] in genus 23, there has been recent focus on showing that Brill–Noether loci of codimension 1 and 2 are distinct, and showing various non-containments of Brill–Noether loci of codimension 2, see [12, 13, 14, 45]; in fact, for $g \geq 34$ and not divisible by 3, one can deduce that there are at least 2 maximal Brill–Noether loci.

These results are proved using a mix of tropical, combinatorial, and limit linear series methods.

On the other hand, our approach is to use K3 surfaces to construct curves admitting a g^r_d, but not a $g^{r'}_{d'}$, thus distinguishing the Brill–Noether loci. This idea was introduced by Farkas [27], and further developed by Lelli-Chiesa [56, 58], who can produce curves on a K3 surface admitting a g^1_d or g^2_d, but not a $g^r_{d'}$. We further extend this technique to curves that admit a g^3_d, which suffices to prove our main theorem.

Theorem 6.2. *Conjecture 6.1 holds in genus* 3–19*,* 22*, and* 23*.*

In genus 23, Eisenbud and Harris [23], and Farkas [27], prove the part of this conjecture concerning the Brill–Noether divisors in their work on the birational geometry of the moduli space of curves. Concerning genus 20 and 21, our results reduce Conjecture 6.1 to the verification that the codimension of $\mathcal{M}^3_{20,17}$ and $\mathcal{M}^4_{21,20}$ is the expected value of 4, and that the codimension of $\mathcal{M}^4_{20,19}$ is at least the expected value of 5, which should be within reach using current techniques.

Section 6.2

Maximal Brill–Noether Loci

In this section, we take a look at the analytic geometry of various Brill–Noether theory conditions on linear systems. We find simple bounds on the maximal Clifford index of Brill–Noether special linear systems and for linear systems that can potentially lift to a K3 surface without contradicting the Hodge index theorem. Furthermore, we find that all non-computing linear systems are always potentially liftable to K3 surfaces. We end with a discussion of how Conjecture 4.6 and lattice theory can imply Conjecture 6.1. We work with a fixed genus g throughout this section.

Let (S, H) be a polarized K3 surface of genus g. In the moduli space $\mathcal{K}^\circ_g$ of polarized K3 surfaces of genus g, the Noether–Lefschetz (NL) locus parameterizes

K3 surfaces with Picard rank > 1. By Hodge theory, the NL locus is a union of countably many irreducible divisors, which we call NL divisors. In [34], Greer, Li, and Tian study the Picard group of $\mathcal{K}_g^\circ$ using Noether–Lefschetz theory and the locus of Brill–Noether special K3 surfaces in $\mathcal{K}_g^\circ$ is identified as a union of NL divisors. More generally, it is convenient to work with the moduli space of primitively quasi-polarized K3 surfaces, denoted $\mathcal{K}_g$ where $\mathcal{K}_g \setminus \mathcal{K}_g^\circ$ is a divisor parameterizing K3 surfaces containing a (-2)-exceptional curve. We define the NL divisor $\mathcal{K}_{g,d}^r$ to be the locus of polarized K3 surfaces $(S, H) \in \mathcal{K}_g$ such that

$$\Lambda_{g,d}^r = \begin{array}{c|cc} & H & L \\ \hline H & 2g-2 & d \\ L & d & 2r-2 \end{array}$$

admits a primitive embedding in $\mathrm{Pic}(S)$ preserving H. In this language, the divisor $\mathcal{K}_g \setminus \mathcal{K}_g^\circ$ is $\mathcal{K}_{g,0}^0$. We note that the $\mathcal{K}_{g,d}^r$ are each irreducible by [68]. As we'll show in Lemma 6.28, polarized K3 surfaces $(S, H) \in \mathcal{K}_{g,d}^r$ should be thought of as those having a curve $C \in |H|$ such that $L \otimes \mathcal{O}_C$ is a line bundle of type g_d^r, and we say that the lattice $\Lambda_{g,d}^r$ is *associated* to g_d^r. Specifically, we have the following lemma, which we prove in Section 6.3.

Lemma 6.3 (See Lemma 6.28). *Let $(S, H) \in \mathcal{K}_{g,d}^r$ and let $C \in |H|$ be a smooth irreducible curve. If L and $H - L$ are basepoint free, $r \geq 2$, and $1 \leq d \leq g-1$, then $L \otimes \mathcal{O}_C$ is a g_d^r.*

Conversely, one is interested in the question of when a given g_d^r on a curve in a K3 surface is the restriction of a line bundle from the K3; in this case, we say that the line bundle is a *lift* of the g_d^r. Lifting of line bundles on curves on K3 surfaces is considered in [21, 32, 56, 57, 61, 75]. In lifting Brill–Noether special linear systems on $C \in |H|$ to a line bundle $L \in \mathrm{Pic}(S)$, we are naturally led to considering two constraints. First, we have $\rho(g,r,d) < 0$ as the linear system is Brill–Noether special. We call the constraint $\rho(g,r,d) < 0$ the *Brill–Noether constraint*. If a g^r_d on a curve $C \in |H|$ on a polarized K3 surface (S,H) has a suitable lift (see Corollary 4.15), then $\mathrm{Pic}(S)$ admits a primitive embedding of $\Lambda^r_{g,d}$ preserving H, and in particular $\mathrm{disc}\big(\Lambda^r_{g,d}\big) < 0$ by the Hodge index theorem. Thus we define

$$\Delta(g,r,d) := \mathrm{disc}\left(\Lambda^r_{g,d}\right) = 4(g-1)(r-1) - d^2 = 4(g-1)(r-1) - (\gamma(r,d)+2r)^2.$$

We thus call the constraint $\Delta(g,r,d) < 0$ the *Hodge constraint* as the inequality stems from the Hodge index theorem. We remark that when $\Delta(g,r,d) < 0$, the Torelli theorem for polarized K3 surfaces implies that a very general K3 surface in $\mathcal{K}^r_{g,d}$ has $\mathrm{Pic}(S) = \Lambda^r_{g,d}$.

Remark 6.4. When considering the lifting of linear systems to K3 surfaces, it is more convenient to consider the Brill–Noether and Hodge constraints for fixed g in the (r,γ)-plane as opposed to the (r,d)-plane, in particular, because the Clifford index of curves on K3 surfaces remains constant in their linear system [32]. In the (r,γ)-plane the Brill–Noether and Hodge constraints determine regions that are bounded by the curves $\rho(g,r,d) = 0$ and $\Delta(g,r,d) = 0$, which we call the *Brill–Noether hyperbola* and *Hodge parabola*, respectively. Simple calculations show that the maximum γ on the Brill–Noether hyperbola is obtained at $r = \sqrt{g} - 1$ and $\gamma = g - 2\sqrt{g} + 1$, the intersection with the line $d = g-1$. Hence, taking $\gamma \leq \lfloor g - 2\sqrt{g} + 1 \rfloor$ suffices to bound Brill–Noether special linear systems. Similarly, the maximum γ on the Hodge parabola is given by $\gamma = \frac{g-5}{2}$, and obtained at the intersection with the line $d = g-1$ at $r = \frac{g+3}{4}$. Thus if $\gamma > \frac{g-5}{2}$ then $\Delta < 0$. Trivially $\lfloor \frac{g-4}{2} \rfloor \geq \frac{g-5}{2}$, and in fact the bound $\gamma \geq \lfloor \frac{g-4}{2} \rfloor \implies \Delta < 0$ is the best possible as seen in genus 9, 13, and 17. As an example, we show the bounds in genus 100, as graphed on the (r,γ)-plane in Section 6.2.

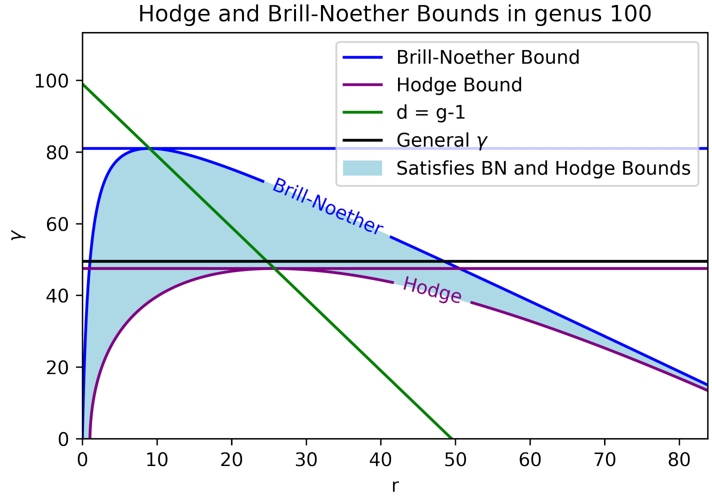

Figure 6.1: The Brill–Noether hyperbola ($\rho = 0$) and the Hodge parabola ($\Delta = 0$) in genus 100. The shaded area satisfies both $\rho < 0$ and $\Delta < 0$.

We recall that the *Clifford index* of a line bundle A on a smooth projective curve C is the integer $\gamma(A) = \deg(A) - 2\,r(A)$ where $r(A) = h^0(C, A) - 1$ is the rank of A. The Clifford index of C is

$$\gamma(C) := \min\{\gamma(A) \mid h^0(C, A) \geq 2 \text{ and } h^1(C, A) \geq 2\}.$$

We say that a line bundle A on C *computes* the Clifford index of C if $\gamma(A) = \gamma(C)$. Clifford's theorem states that $0 \leq \gamma(C) \leq \lfloor \frac{g-1}{2} \rfloor$, and when C is a general curve of genus g, $\gamma(C) = \lfloor \frac{g-1}{2} \rfloor$.

Definition 6.5. Let A be a Brill–Noether special g^r_d on a curve C of genus g, i.e. $\rho(g, r, d) < 0$. We say A is *non-computing* if $\gamma(r, d) > \lfloor \frac{g-1}{2} \rfloor$, that is, A is a Brill–Noether special g^r_d that cannot compute the Clifford index of C.

Lemma 6.6. *Let $g \geq 14$, $r \geq 2$, and $2r \leq d \leq g - 1$. If $\mathcal{M}^r_{g,d}$ is an expected maximal Brill–Noether locus, then $\gamma(d, r) = d - 2r > \lfloor \frac{g-1}{2} \rfloor$. When $g < 14$, there are no non-computing g^r_d's.*

Proof. One can easily check that if $d-2r \leq \lfloor \frac{g-1}{2} \rfloor$, then $\rho(g,r,d+1) < 0$, and hence $\mathcal{M}^r_{g,d}$ is not an expected maximal Brill–Noether locus. When $g < 14$, this is a simple computation enumerating all g^r_d's with Clifford index $\leq \lfloor \frac{g-1}{2} \rfloor + 1$. □

Thus for genus $g \geq 14$, except for $\mathcal{M}^1_{g,\lfloor \frac{g+1}{2} \rfloor}$, all the maximal Brill–Noether loci are those associated to non-computing g^r_ds. If lifting results are able to distinguish between maximal Brill–Noether loci, there should not be an obvious obstruction to lifting the associated linear systems. In particular, the Hodge index theorem implies that the lattices obtained by lifting should have negative discriminant, which we show is true for non-computing g^r_ds below.

Proposition 6.7. *Let g, r, d be natural numbers with $2 \leq d \leq g-1$ and $1 \leq r \leq g-1$. Then the Hodge parabola lies under the Brill–Noether hyperbola. In particular, all non-computing linear systems, and all expected maximal Brill–Noether loci, satisfy $\Delta < 0$.*

Proof. For fixed $g \geq 2$, and for each constraint ($\rho = 0$ or $\Delta = 0$), we solve for γ as a function of r and g. For $\rho(g,r,\gamma) = 0$, we find $\gamma_\rho(r) = g - r - \frac{g}{r+1}$. Likewise for $\Delta(g,r,\gamma) = 0$ we have $\gamma_\Delta(r) = 2\sqrt{(g-1)(r-1)} - 2r$. Observe that $\gamma_\rho = \gamma_\Delta$ has no solutions in the given range (solve for r in terms of g, and note that $g \geq 2$). Finally, since $\gamma_\rho(1) > 0$ and $\gamma_\Delta(1) < 0$, we see by continuity that $\gamma_\rho(r) - \gamma_\Delta(r) > 0$.

The bound $\gamma \geq \lfloor \frac{g-4}{2} \rfloor$ implies that $\Delta < 0$, as in the remark above. Since this is below the general Clifford index ($\lfloor \frac{g-1}{2} \rfloor$), we see that any lattice associated to a non-computing linear system will have negative discriminant. In particular, by Lemma 6.6 above, this applies to the expected maximal linear systems. □

We thus conjecture (Conjecture 6.1) that the maximal Brill–Noether loci are exactly the *expected maximal Brill–Noether loci*, which are Brill–Noether loci $\mathcal{M}^r_{g,d}$ where for fixed r, d is maximal such that $\rho(g,r,d) < 0$ and $\rho(g,r-1,d-1) \geq 0$. Equivalently, the expected maximal Brill–Noether loci correspond to the maximal g^r_d lying under the Brill–Noether hyperbola for each r, up to the containments $\mathcal{M}^r_{g,d} \subseteq \mathcal{M}^r_{g,d+1}$ when $\rho(g,r,d+1) < 0$ and $\mathcal{M}^r_{g,d} \subset \mathcal{M}^{r-1}_{g,d-1}$ when $\rho(g,r-1,d-1) < 0$.

One could imagine that if there are any unexpected containments among Brill–Noether loci, then some would come from containments of the form $\mathcal{M}^1_{g,d} \subset \mathcal{M}^r_{g,d'}$. However, we find that the expected maximal $\mathcal{M}^1_{g,d}$ is not contained in the other expected maximal loci.

Proposition 6.8. *Let $\rho(g,r,d) < 0$, and $\gamma(r,d) \geq \lfloor \frac{g-1}{2} \rfloor + 1$, e.g., for a non-computing g^r_d. Then $\mathcal{M}^1_{g,\lfloor \frac{g+1}{2} \rfloor} \not\subseteq \mathcal{M}^r_{g,d}$. When $9 \leq g < 14$, $\mathcal{M}^1_{g,\lfloor \frac{g+1}{2} \rfloor} \not\subseteq \mathcal{M}^r_{g,d}$ for an expected maximal Brill–Noether locus with $r \geq 2$.*

Proof. Let $k = \frac{g+1}{2}$, and $r' = \min\{r, g-d+r-1\}$. We compute

$$\begin{aligned}
\rho_k &= \max_{\ell \in \{0,\dots,r'\}} \rho(g, r-\ell, d) - \ell k = \rho(g,r,d) + (g - k - \gamma(r,d) + 1)\ell - \ell^2 \\
&\leq \max_{\ell \in \{0,\dots,r'\}} \rho(g,r,d) + \ell \left(g - \left\lfloor \frac{g-1}{2} \right\rfloor - \left\lfloor \frac{g+1}{2} \right\rfloor \right) - \ell^2 \\
&< \max_{\ell \in \{0,\dots,r'\}} \rho(g,r,d) + 2\ell - \ell^2 \leq \rho(g,r,d) + 1 \leq 0.
\end{aligned}$$

Therefore $\rho_k < 0$. From [70, Theorem 1.1], as $\dim W^r_d(C) \leq \rho_k$, and $W^r_d(C)$ is empty if its dimension is negative, we see that a general k-gonal curve does not admit a g^r_d. Hence $\mathcal{M}^1_{g,\lfloor \frac{g+1}{2} \rfloor} \not\subseteq \mathcal{M}^r_{g,d}$.

The statement for $9 \leq g < 14$ is obtained simply by calculating ρ_k explicitly, and noting that in each case $\rho_k < 0$. □

Remark 6.9. In [27], Farkas asks the general question of when does a general k-gonal curve of genus g have no other linear series g^r_d with $\rho(g,r,d) < 0$? The above proposition answers the case when $k = \lfloor \frac{g+1}{2} \rfloor$, when the curve has maximal sub-general gonality. If a curve has a Brill–Noether special $g^{r'}_{d'}$, then it has a g^r_d for an expected maximal Brill–Noether locus, and the above shows this is not the case. In general, this question is answered by recent breakthroughs in Brill–Noether theory for curves of fixed gonality, see e.g. [16, 27, 43, 51, 52, 69, 70].

In Lemma 6.28, we show that under mild assumptions, the curves $C \in |H|$ on a polarized K3 surface (S, H) with $\mathrm{Pic}(S) = \Lambda^r_{g,d}$ associated to an expected maximal locus with $r \geq 2$, all have general Clifford index. Thus the $\mathcal{M}^1_{g,\lfloor \frac{g+1}{2} \rfloor}$ does not contain other expected maximal loci in many genera. Similar results have been proven by Farkas and Lelli-Chiesa [27, 56].

A natural question is whether lattices corresponding to g^r_ds can be contained as sublattices in each other. In general, the answer is yes. Already in genus 14, we see that $\Lambda^2_{14,10}$ could be embedded as a sublattice of $\Lambda^2_{14,8}$. However, these are not associated to expected maximal loci. In particular, we would like to show that lattices associated to expected maximal loci cannot contain any lattices associated to other g^r_d. This turns out to be false (see Section 6.2.1). However, we can prove that lattices associated to Brill–Noether special linear systems with lower than general Clifford index cannot be contain in lattices associated to expected maximal loci, and that any containments between lattices associated to an expected maximal loci and those associated to non-computing g^r_ds must be equalities.

Proposition 6.10. *Let $\Lambda^r_{g,d}$ be associated to an expected maximal g^r_d.*

(i) Any lattice $\Lambda^{r'}_{g,d'}$ associated to a special $g^{r'}_{d'}$ with $\gamma(g^{r'}_{d'}) < \lfloor \frac{g-1}{2} \rfloor$ for any r' or $\gamma(g^{r'}_{d'}) = \lfloor \frac{g-1}{2} \rfloor$ if $r' \neq 1$ cannot be contained in $\Lambda^r_{g,d}$.

(ii) Let $d' \leq g-1$. Any lattice $\Lambda^{r'}_{g,d'}$ associated to another expected maximal $g^{r'}_{d'}$ is not contained in $\Lambda^r_{g,d}$, unless the lattices are isomorphic. Similarly, any lattice associated to a non-computing $g^{r'}_{d'}$ with $d' \leq g-1$ is not contained in the lattice associated to an expected maximal g^r_d unless they are isomorphic.

Proof. To simplify notation, we write Δ for the discriminant of a lattice Λ.

To prove (i), we recall that if $\Lambda_{sub} \subset \Lambda_{exp}$ is a finite index sublattice, then we have $\Delta_{sub} = [\Lambda_{exp} : \Lambda_{sub}]^2 \Delta_{exp}$. We calculate that the ratio $\frac{\Delta_{sub}}{\Delta_{exp}}$ is never a square for the lattices considered. Specifically, we show that the largest negative discriminant $-\Delta_{sub}$ among lattices with $\gamma < \lfloor \frac{g-1}{2} \rfloor$, divided by the negative discriminant $-\Delta_{exp}$ of any lattice associated to an expected maximal linear system, is not an integer. Because $\Delta(g,r,d) = \mathrm{disc}(-\Lambda^r_{g,d}) = d^2 - 4(g-1)(r-1)$, it is clear that for fixed γ this decreases as r increases until $d = g-1$. It follows that none of the lattices considered can be contained in $\Lambda^1_{g,\lfloor \frac{g+1}{2} \rfloor}$, the expected maximal loci with $r = 1$. From now on, we assume $r > 1$. Furthermore, we can take

- $\max(-\Delta_{sub}) = d^2$ with $d = \frac{g+1}{2}$ when $\gamma = \frac{g-1}{2} - 1$; or
- $\max(-\Delta_{sub}) = d^2 - 4(g-1)$ with $d = \frac{2}{3}g + 2$ when $\gamma = \frac{g-1}{2}$.

We also note that $-\Delta$ increases when r and γ both increase by 1, and increases as γ increases for fixed r. Thus if $r' \geq r$, then clearly $\frac{\max(-\Delta_{sub})}{-\Delta_{exp}} < 1$. If $r' < r$, then moving from $g^{r'}_{d'}$ to g^r_d, we take steps increasing r' and γ by 1 until we hit r (and then take steps increasing γ) or hit the line $d = g-1$ and we take steps increasing γ by 1 and decreasing r' by 1. Since each of these steps increase $-\Delta$, we again see that $\frac{\max(-\Delta_{sub})}{-\Delta_{exp}} < 1$. We can always take these steps since we may assume we start at $r = 1$ or $r = 2$, and the expected maximal g^r_d lie far above. Thus (i) is proved.

To prove (ii), we similarly bound $\max(-\Delta)$ and $\min(-\Delta)$ for non-computing g^r_ds. It can be verified that the ratio $\frac{\min(-\Delta)}{\max(-\Delta)} > \frac{1}{4}$ for $r < \sqrt{g}$, and hence $\max(-\Delta) < 4\min(-\Delta)$, thus the discriminants of lattices associated to the expected maximal Brill–Noether loci cannot differ by a square greater than 1. Hence if the lattices associated to expected maximal loci are contained, they must be the same lattice. Since $-\Delta$ increases as r decreases and as γ increases until $d = g-1$, this argument in fact shows that any lattice associated to a non-expected maximal non-computing

$g^{r'}_{d'}$ cannot be contained in the lattice of an expected maximal g^r_d unless they have the same discriminant. $\square$

Remark 6.11. In fact, computation up to large genus shows that the lattices associated to expected maximal loci do not contain any lattices associated to other expected maximal loci. We conjecture that this is always true, though a proof of this is currently unknown.

6.2.1. Program: Donagi–Morrison implies maximal Brill–Noether loci

To verify Conjecture 6.1, our strategy is for fixed genus g and distinct expected maximal $\mathcal{M}^r_{g,d}$ and $\mathcal{M}^{r'}_{g,d'}$ to prove that for a very general K3 surface $(S, H) \in \mathcal{K}^r_{g,d}$, a smooth curve $C \in |H|$ admits a g^r_d but not a $g^{r'}_{d'}$. We do this by combining three kinds of results: (i) a lifting result, (ii) showing that $C \in |H|$ has a g^r_d given by restricting $L \in \Lambda^r_{g,d}$, and (iii) a comparison result that distinguishes lattices. The latter two can be checked for any fixed genus. If all the lattices can be distinguished, a lifting result like the Donagi–Morrison conjecture (Conjecture 4.6) implies Conjecture 6.1.

We start by defining a few terms in Conjecture 4.6.

Definition 6.12. Let S be a K3 surface, $C \subset S$ be a curve, and $A \in \operatorname{Pic}(C)$ and $M \in \operatorname{Pic}(S)$ be line bundles. We say that the linear system $|A|$ is contained in the restriction of $|M|$ to C when for every $D_0 \in |A|$, there is some divisor $M_0 \in |M|$ such that $D_0 \subset C \cap M_0$.

Definition 6.13. A line bundle M is *adapted* to $|H|$ when

(i) $h^0(S, M) \geq 2$ and $h^0(S, H \otimes M^\vee) \geq 2$; and

(ii) $h^0(S, M \otimes \mathcal{O}_C)$ is independent of the smooth curve $C \in |H|$.

Thus whenever M is adapted to $|H|$, condition (i) ensures that $M \otimes \mathcal{O}_C$ contributes to $\gamma(C)$, and condition (ii) ensures that $\gamma(M \otimes \mathcal{O}_C)$ is constant as C varies in its linear system and is satisfied if either $h^1(S, M) = 0$ or $h^1(S, H \otimes M^\vee) = 0$.

For convenience, we recall the notion of a Donagi–Morrison lift, Definition 4.7.

Definition 6.14. Let (S, H) be a polarized K3 surface and $C \in |H|$ be a smooth irreducible curve of genus ≥ 2. Suppose A is a complete basepoint free g^r_d on C such that $d \leq g - 1$ and $\rho(g, r, d) < 0$. We call a line bundle M a *Donagi–Morrison lift* of A if M satisfies the conditions in Conjecture 4.6. That is,

- M is adapted to $|H|$,
- $|A|$ is contained in the restriction of $|M|$ to C, and
- $\gamma(M \otimes \mathcal{O}_C) \leq \gamma(A)$.

We call a line bundle M a *potential Donagi–Morrison lift* of A if M satisfies $\gamma(M \otimes \mathcal{O}_C) \leq \gamma(A)$ and $d(M \otimes \mathcal{O}_C) \geq d(A)$. Note that a Donagi–Morrison lift is a potential Donagi–Morrison lift. We say a (potential) Donagi–Morrison lift is of type g^s_e if $M^2 = 2s - 2$ and $M.H = e$.

We summarize a few potential results distinguishing lattices, each of which would be useful in verifying Conjecture 6.1 given an appropriate lifting result.

(L1) For a fixed lattice $\Lambda^r_{g,d}$ associated to an expected maximal $\mathcal{M}^r_{g,d}$ and any lattice $\Lambda^{r'}_{g,d'}$ associated to another expected maximal $\mathcal{M}^{r'}_{g,d'}$, one has $\Lambda^{r'}_{g,d'} \nsubseteq \Lambda^r_{g,d}$.

(L2) For a fixed lattice $\Lambda^r_{g,d}$ associated to an expected maximal $\mathcal{M}^r_{g,d}$ and any lattice $\Lambda^{r'}_{g,d'}$ with $\lfloor \frac{g+1}{2} \rfloor \leq \gamma(r', d') \leq \lfloor g - 2\sqrt{g} + 1 \rfloor$ and $1 \leq r' \leq \lfloor \frac{g-1-\gamma(r',d')}{2} \rfloor$, one has $\Lambda^{r'}_{g,d'} \nsubseteq \Lambda^r_{g,d}$.

(L3) For a pair of lattices $(\Lambda^r_{g,d}, \Lambda^{r'}_{g,d'})$ both associated to expected maximal Brill–Noether loci, and any lattice $\Lambda^s_{g,e}$ such that $\lfloor \frac{g+1}{2} \rfloor \leq \gamma(s, e) \leq \gamma(r', d')$ and $1 \leq s \leq \lfloor \frac{g-1-\gamma(s,e)}{2} \rfloor$, one has $\Lambda^s_{g,e} \nsubseteq \Lambda^r_{g,d}$.

We note that (**L2**) implies (**L1**). Furthermore, for fixed r and d, (**L2**) implies (**L3**) for all r' and d'.

Remark 6.15. The bounds on $\gamma(s,e)$ and s in (**L3**) include all lattices associated to a potential Donagi–Morrison lift of a $g^{r'}_{d'}$. Indeed, suppose M is a potential Donagi–Morrison lift of a $g^{r'}_{d'}$, and say M is of type g^s_e. The lower bound on $\gamma(s,e)$ comes from Proposition 6.10 (i). Since M is a potential Donagi–Morrison lift of a $g^{r'}_{d'}$, we have $\gamma(s,e) \leq \gamma(r',d')$, which is the upper bound on $\gamma(s,e)$. Since $M \otimes \mathcal{O}_C$ contributes to $\gamma(C)$, this forces $H \otimes M^\vee \otimes \mathcal{O}_C$ to be at least a g^1_{2g-2-e}, whereby $s \leq \frac{g-1-\gamma(s,e)}{2}$ as $2s \leq e$, which gives the upper bound on s.

Similarly, the bounds in (**L2**) include all lattices associated to a potential Donagi–Morrison lift of an expected maximal linear system. $M \otimes \mathcal{O}_C$ must have Clifford index no bigger than the expected maximal g^r_d by Conjecture 4.6, the upper bound on $\gamma(r',d')$ comes from Remark 6.4. The other bounds are obtained in the same way as for (**L3**).

Remark 6.16. As stated above, computations show that (**L1**) holds for every expected maximal locus up to large genus.

We note that (**L2**) and (**L3**) do not always hold. The first genus where (**L3**) fails is $g = 56$, where (**L3**) fails for the lattices $\Lambda^r_{g,d} = \Lambda^2_{56,39}$ and $\Lambda^{r'}_{g,d'} = \Lambda^6_{56,49}$; indeed, in attempting to check whether $\mathcal{M}^2_{56,39}$ can be contained in $\mathcal{M}^3_{56,44}$, a g^3_{44} on a curve $C \in |H|$ for a very general $(S,H) \in \mathcal{K}^2_{56,39}$ has a potential Donagi–Morrison lift M of type g^6_{49}. However, $\Lambda^2_{56,39} \cong \Lambda^6_{56,49}$, and so (**L3**) does not hold. In this case, because $\rho(56,2,39) = -1$ and $\rho(56,3,44) = -4$, we clearly have $\mathcal{M}^2_{56,39} \nsubseteq \mathcal{M}^3_{56,44}$. Hence the failure of (**L3**) does not necessarily obstruct our program to prove that Conjecture 4.6 implies Conjecture 6.1.

The next genus where (**L3**) fails is $g = 89$, where the locus $\mathcal{M}^3_{89,69}$ could possibly be contained in $\mathcal{M}^4_{89,75}$ or $\mathcal{M}^5_{89,79}$. This is because line bundles of type g^3_{36} and

g^4_{75} have a potential Donagi–Morrison lift M of type g^{10}_{85}, and the lattice $\langle H, M\rangle = \Lambda^{10}_{89,85}$ is isomorphic to $\Lambda^3_{89,69}$, so that (**L3**) does not hold. In this example, $\mathcal{M}^3_{89,69}$ has codimension 3 in $\mathcal{M}_{89}$, whereas $\mathcal{M}^4_{89,75}$ and $\mathcal{M}^5_{89,79}$ both have codimension 1, hence the codimensions of the loci do not rule out the possibility that $\mathcal{M}^3_{89,69}$ is not maximal. Thus in genus 89, Conjecture 4.6 together with (**L2**) is not sufficient to imply Conjecture 6.1 without additional techniques.

We note that below genus 200, except for genus $56, 89, 91, 92, 145, 153$, and 190, (**L2**) holds, and thus Conjecture 4.6 implies Conjecture 6.1.

Proposition 6.17. *Let $\mathcal{M}^r_{g,d}$ and $\mathcal{M}^{r'}_{g,d'}$ be two expected maximal Brill–Noether loci. Suppose (S, H) is a polarized K3 surface with $\mathrm{Pic}(S) = \Lambda^r_{g,d}$, and $L \otimes \mathcal{O}_C$ is a g^r_d. If the Donagi–Morrison conjecture (Conjecture 4.6) holds for $g^{r'}_{d'}$ on C and (**L3**) holds for the pair $(\Lambda^r_{g,d}, \Lambda^{r'}_{g,d'})$, then $\mathcal{M}^r_{g,d} \nsubseteq \mathcal{M}^{r'}_{g,d'}$. In particular, if Conjecture 4.6 and (**L2**) hold for all expected maximal g^r_d in genus g, then Conjecture 6.1 holds in genus g.*

Proof. The condition (**L3**) implies that $\mathrm{Pic}(S)$ cannot admit any potential Donagi–Morrison lift of the $g^{r'}_{d'}$. Hence the existence of a $g^{r'}_{d'}$ on C contradicts the Donagi–Morrison conjecture. Therefore C has no $g^{r'}_{d'}$, as was to be shown. □

To state a related question, we need a simple definition.

Definition 6.18. For a Brill–Noether special curve C, we define the *special Clifford index* of C as

$$\widetilde{\gamma}(C) := \min\{\gamma(A) \mid \rho(A) < 0,\ h^0(C, A) \geq 2, \text{ and } h^1(C, A) \geq 2\}.$$

We say a Brill–Noether special line bundle A on C *computes* the special Clifford index if $\gamma(A) = \widetilde{\gamma}(C)$.

Lelli-Chiesa's lifting result [57, Theorem 4.2] provides a lift of Brill–Noether special line bundles computing the Clifford index. A similar result for line bundles computing the special Clifford index of the curve together with L1 would imply that $\mathcal{M}^r_{g,d} \not\subseteq \mathcal{M}^{r'}_{g,d'}$ for $\gamma(r,d) \geq \gamma(r',d')$. We are left with three questions to which positive answers would imply parts of Conjecture 6.1.

Question 6.19. When (**L1**) or (**L2**) fail, can the Brill–Noether loci be distinguished in another way?

Question 6.20. Under what conditions does a line bundle computing the special Clifford index of a curve C lift to a line bundle on S?

Question 6.21. Does the Donagi–Morrison conjecture hold for expected maximal g^r_ds?

We note that the work on Brill–Noether theory for fixed gonality, if it were extended to higher rank, could provide another approach to distinguishing Brill–Noether loci that is complementary to the Donagi–Morrison lifting approach.

Section 6.3

Maximal Brill–Noether Loci in Genus ≤ 23

In this section, we identify the maximal Brill–Noether loci in genus 3–19, 22, and 23, proving Theorem 6.2. Our technique combines known results about non-containments of Brill–Noether loci, work by Lelli-Chiesa [56] on lifting of rank 2 linear systems and linear systems computing the Clifford index, together with our lifting results for rank 3 linear systems above.

6.3.1. Genus 3–6

By Clifford's theorem, any Brill–Noether special curve of genus 3 or 4 is hyperelliptic, hence $\mathcal{M}^1_{g,2}$ is the only maximal (and expected maximal) Brill–Noether locus. Similarly, in genus 5, every Brill–Noether special curve has gonality ≤ 3, hence $\mathcal{M}^1_{5,3}$ is the only maximal (and expected maximal), Brill–Noether locus. Thus Conjecture 6.1 holds in genus 3–5. In genus 6, we verify the conjecture as well.

Proposition 6.22. *The maximal Brill–Noether loci in genus* 6 *are* $\mathcal{M}^1_{6,3}$ *and* $\mathcal{M}^2_{6,5}$.

Proof. $\mathcal{M}^1_{6,3}$ and $\mathcal{M}^2_{6,5}$ are the expected maximal Brill–Noether loci. It remains to show that they are distinct. Since $\rho(6,1,3) = -2$ and $\rho(6,2,5) = -3$, results on the codimension of Brill–Noether loci (e.g., [22, 24, 78]) imply that $\mathcal{M}^1_{6,3} \not\subseteq \mathcal{M}^2_{6,5}$. A smooth plane quintic curve C has genus 6. By a well-known result of Max Noether [40], C has gonality 4, hence has no g^1_3. Thus $\mathcal{M}^2_{6,5} \not\subseteq \mathcal{M}^1_{6,3}$. ∎

6.3.2. Unexpected containments in genus 7–9

In each genus 7-9, there are two expected maximal Brill–Noether loci, and we give detailed constructions of the unexpected containments between them: These are $\mathcal{M}^2_{7,6} \subset \mathcal{M}^1_{7,4}$, $\mathcal{M}^1_{8,4} \subset \mathcal{M}^2_{8,7}$, and $\mathcal{M}^2_{9,7} \subset \mathcal{M}^1_{9,5}$. Thus in these genera, there is a unique maximal Brill–Noether locus. In genus 7 and 9, we are indebted to Hannah Larson for pointing them out.

Proposition 6.23. *Every Brill–Noether special curve of genus* 7 *has a* g^1_4.

Proof. The expected maximal Brill–Noether loci in genus 7 are $\mathcal{M}^1_{7,4}$ and $\mathcal{M}^2_{7,6}$. We show that every smooth genus 7 curve with a g^2_6 has a g^1_4. Let $\phi : C \to \mathbb{P}^2$ be the map given by the g^2_6. If the g^2_6 is not very ample (i.e. if the induced map is not birational), then subtracting any two general points on C gives a g^1_4 (see [38, Chapter IV, Proposition 3.1]). Thus we can assume ϕ is a nondegenerate map, so that $\phi(C)$ is a plane curve of degree 6, so has arithmetic genus 10. Hence $\phi(C)$ must have a singular point (a point of multiplicity ≥ 2). Projecting from this point gives a g^1_k for $k \leq 4$, hence a g^1_4. ∎

Proposition 6.24 (Mukai [63, Lemma 3.8])**.** *Every Brill–Noether special curve of genus* 8 *has a* g^2_7.

Proof. The maximal Brill–Noether loci in genus 8 are $\mathcal{M}^1_{8,4}$ and $\mathcal{M}^2_{8,7}$. We show that a curve C of genus 8 with a g^1_4 has a g^2_7. Let A be a line bundle of type g^1_4 on C. If C has a g^2_6 then it has a g^2_7, thus we may assume that C has no g^2_6, hence no g^3_8 (Serre adjoint to a g^2_6). Similarly, we can assume C has no g^1_3 (as twice a g^1_3 is a g^2_6), whence $|A|$ is basepoint free. Furthermore, the Serre adjoint A' of A is of type g^4_{10} and is very ample as there is no g^3_8. Hence $|A'|$ exhibits C as degree 10 curve in $\mathbb{P}^4$. This embedding of C has 8 trisecant lines by the Berzolari formula

$$\#\{\text{trisecant lines to } C\} = \frac{(d-2)(d-3)(d-4)}{6} - g(d-4),$$

where g is the genus of C and d is the degree of C in $\mathbb{P}^4$, see [8]. Projecting from one of the trisecant lines gives a g^2_7. □

Proposition 6.25. *Every Brill–Noether special curve of genus 9 has a g^1_5.*

Proof. The expected maximal Brill–Noether loci in genus 9 are $\mathcal{M}^1_{9,5}$ and $\mathcal{M}^2_{9,7}$. We will show, similarly to Proposition 6.23, that every smooth genus 9 curve with a g^2_7 has a g^1_5. Let $\phi : C \to \mathbb{P}^2$ be the map given by the g^2_7. If the g^2_7 is not very ample, then C has a g^1_5. Thus we can assume ϕ is a nondegenerate map, so that $\phi(C)$ is a plane curve of degree 7, so has arithmetic genus 15. Hence $\phi(C)$ must have a singular point of multiplicity ≥ 2. Projecting from this point gives a g^1_k for $k \leq 5$, hence a g^1_5. □

Remark 6.26. The constructions in genus 7–9 rely on projections from secant linear spaces. Given a very ample linear system of type g^r_d defining an embedding $C \to \mathbb{P}^r$ of degree d, if C admits a k-secant l-dimensional linear subspace of $\mathbb{P}^r$, then projection from that linear subspace results in a g^{r-l-1}_{d-k}. The expected dimension of the space of l-dimensional linear spaces of $\mathbb{P}^r$ that are k-secant to C is classically known to be $k-(k-l-1)(r-l)$, see [29]. Secant linear spaces for which this expected dimension is nonnegative (resp. negative) are called *expected* (resp. *unexpected*). When the expected dimension is 0, there are unwieldy enumerative formulas for the expected number of such secant linear spaces generalizing the Berzolari formula, see [4, VIII.4]. We have checked that the only cases when an expected maximal g^r_d (or its Serre adjoint) admits an expected k-secant l-dimensional linear space and such that the associated g^{r-l-1}_{d-k} is also Brill–Noether special (and not Serre adjoint to a g^r_d) are the three cases discussed above in genus 7–9. Thus no additional unexpected containments of expected maximal Brill–Noether loci can arise from expected secant linear spaces. Unexpected secant linear spaces could potentially give rise to other unexpected containments, but these should not exist if we believe various versions of the Donagi–Morrison conjecture for expected maximal Brill–Noether special linear systems, see [57, Theorem 1.4].

6.3.3. Genus 10–13

We first establish a few useful lemmas which, in effect, say that if $\mathrm{Pic}(S) = \langle H, L\rangle$ looks like it is obtained by lifting a g^r_d on $C \in |H|$ to a line bundle L, then L is in fact a lift of a g^r_d. Moreover, for these lifts, we would like the line bundle to be basepoint free, which is true if the g^r_d is primitive. In particular, our next lemma shows that if a curve C on a K3 surface strictly contains a Brill–Noether special linear system, then it is primitive.

Lemma 6.27. *Let (S, H) be a polarized K3 surface of genus g, $C \in |H|$ a smooth connected curve, and $A \in \mathrm{Pic}(C)$ be a line bundle of type g^r_d. Suppose that $\rho(g, r, d) < 0$ and C has no Brill–Noether special linear series of Clifford index smaller than A. Then A is primitive.*

Proof. We note that $\gamma(\omega_C \otimes A^\vee) = \gamma(A)$, $\rho(A) = \rho(\omega_C \otimes A^\vee)$, $\gamma(A - P) < \gamma(A)$ when P is a basepoint of A, and $\rho(g, r, d-1) < \rho(g, r, d)$. Suppose A has a basepoint

P. Then $A - P$ has strictly smaller Clifford index and is Brill–Noether special. By assumption, C cannot be in the linear series $|A - P|$. Thus A is basepoint free. Likewise, if $\omega_C \otimes A^\vee$ has a basepoint P, then $\omega_C \otimes A^\vee - P$ is Brill–Noether special and has smaller Clifford index, which cannot be the case. □

Parts of the following Lemma go back to Farkas in [27] and Rathmann's Theorem (see [47, 74]).

Lemma 6.28. *Let (S, H) be a polarized K3 surface of genus g in the Noether–Lefschetz divisor $\mathcal{K}^r_{g,d}$, i.e., with $\mathrm{Pic}(S)$ admitting a primitive embedding of the sublattice*

$$\Lambda^r_{g,d} = \begin{array}{c|cc} & H & L \\ \hline H & 2g-2 & d \\ L & d & 2r-2 \end{array}$$

Let $C \in |H|$ be a smooth irreducible curve.

(i) If $\mathrm{Pic}(S) = \Lambda^r_{g,d}$ and $2 \leq r, d \leq g-1$, then L is nef.

(ii) If L and $H - L$ are basepoint free, $r \geq 2$, and $0 < d \leq g-1$, then $L \otimes \mathcal{O}_C$ is a g^r_d. (The assumption on basepoint free-ness is achieved if for example S has no (-2)-curves, or can be checked numerically.)

(iii) Suppose that $L \otimes \mathcal{O}_C$ is a g^r_d with $\gamma(r, d) > \lfloor \frac{g-1}{2} \rfloor$ and $\rho(g, r, d) < 0$ and that all lattices obtained by lifting special linear systems of general Clifford index or lower cannot be contained in $\mathrm{Pic}(S)$. Then C has Clifford index $\gamma(C) = \lfloor \frac{g-1}{2} \rfloor$, maximal gonality $\lfloor \frac{g+3}{2} \rfloor$, and Clifford dimension 1.

(iv) If $\mathrm{Pic}(S) = \Lambda^r_{g,d}$ is associated to an expected maximal g^r_d, then the assumption on lattices in (iii) holds.

(v) Suppose that $\gamma(r,d) \leq \lfloor \frac{g-1}{2} \rfloor$, $\rho(g,r,d) < 0$, and that all lattices obtained by lifting special linear systems A not of type g^r_d with $\gamma(A) \leq \lfloor \frac{g-1}{2} \rfloor$ cannot be contained in $\mathrm{Pic}(S)$. Then $L \otimes \mathcal{O}_C$ is a g^r_d and $\gamma(C) = \gamma(r,d)$.

Proof. To prove (i) we show that for any (-2)-curve $\Gamma = aH + bL \in \Lambda^r_{g,d}$, we have $\Gamma.L \geq 0$. We note that as Γ is a (-2)-curve, a and b must have opposite sign. We prove (i) in three cases.

First suppose $a > 0$ and $b < 0$. Then as $\Gamma.H \geq 1$ and $a > 0$, we have $b\Gamma.L \leq -2$, thus as $b < 0$, $\Gamma.L \geq 0$.

Second, suppose $a < -1$ and $b > 0$. Then since $\Gamma.H \geq 1$, we have $a\Gamma.H \leq -2$. Thus $b.\Gamma.L \geq 0$, and since $b > 0$ we must have $\Gamma.L \geq 0$.

Lastly, suppose $a = -1$ and $b > 0$. We see that if $\Gamma.H \geq 2$, then we can follow the same argument as above to see that L is nef. Thus the only remaining case is when $a = -1$ and $\Gamma.H = 1$. We calculate $2g - 2 = (H + \Gamma)^2 = (bL)^2 = b^2(2r - 2)$, hence $b^2 = \frac{g-1}{r-1} \in \mathbb{Z}$. From $\Gamma.H = 1$, we see $b = \frac{2g-1}{d}$, and plugging this in to $2g - 2 = b^2(2r - 2)$ yields

$$d^2(g-1) = (2g-1)^2(r-1).$$

Looking modulo $g - 1$, we immediately see that $r - 1 \equiv 0 \mod g - 1$, hence $\frac{r-1}{g-1} \in \mathbb{Z}$, and thus $r = g$, which is a contradiction. Thus L is always nef.

To prove (ii), we note that L is clearly a lift of a $g^{r'}_d$ on C for some $r' \geq 0$. Since $0 < d \leq g - 1$, we see that $L^2, (H - L)^2 > 0$. Furthermore, since $H.L, H.(H - L) > 0$, both these line bundles are nontrivial and intersect H positively, hence $h^0(S, L), h^0(S, H - L) \geq 2$. By assumption, L and $H - L$ are basepoint free, and thus globally generated. Therefore Corollary 4.15 applies. Thus, as $L^2 = 2r - 2$, we see that $L \otimes \mathcal{O}_C$ must be a divisor of type g^r_d. Hence (ii) is proved.

To prove (iii), we note that a $g^1_{d'}$ with $\rho(g, 1, d') < 0$ has Clifford index $\gamma(g^1_{d'}) <$

$\lfloor \frac{g-1}{2} \rfloor$. Suppose for contradiction that C has lower than general Clifford index. Then by [57, Theorem 4.2] we would be able to lift some special linear system computing $\gamma(C)$ to a divisor $L' \in \mathrm{Pic}(S)$, and by assumption $\langle H, L' \rangle$ cannot be contained in $\mathrm{Pic}(S)$. Thus C has general Clifford index. The same argument shows that C cannot have a special linear system computing its Clifford index. Thus C has a $g^1_{\lfloor \frac{g+3}{2} \rfloor}$ which computes the Clifford index. Hence C has maximal gonality and Clifford dimension 1.

To prove (iv), we note that if C had any Brill–Noether special $g^{r'}_{d'}$ with $\gamma(g^{r'}_{d'}) \leq \frac{g-1}{2}$, then it has a g^r_d with $\gamma = \frac{g-1}{2}$ or a g^1_d with $\gamma(g^1_d) = \frac{g-1}{2} - 1$. Thus we only need to consider lattices $\Lambda^r_{g,d}$ associated to those g^r_d. The proof is now Proposition 6.10(i). Thus (iv) is proved.

To prove (v), we note again that $L \otimes \mathcal{O}_C$ is a $g^{r'}_d$. If $r' \neq r$, then $\gamma(C) \neq \gamma(r,d)$ and some line bundle A would compute $\gamma(C)$. Thus there would exists some lift of A to a line bundle L', but again the lattice $\langle H, L' \rangle \nsubseteq \mathrm{Pic}(S)$. Hence $r' = r$ and we see that $L \otimes \mathcal{O}_C$ is a g^r_d. Similarly, $\gamma(C) = \gamma(r,d)$. □

Remark 6.29. If $\Lambda^r_{g,d}$ has a (-2)-curve, there are still some ways to check that L and $H - L$ are basepoint free. Namely, if they are both nef, then we can check they are basepoint free by checking if there are any elliptic curves on S. Namely if $N \in \mathrm{Pic}(S)$ is nef and there are no elliptic curves, then N is basepoint free by a well-known result of Saint-Donat. To numerically check if $D \in \mathrm{Pic}(S)$ is nef, one can check whether $D.\Gamma \geq 0$ for any (-2)-curve Γ.

One can also check that $L \otimes \mathcal{O}_C$ is a g^r_d by enumerating all of the degree d $g^{r'}_d$ on C and using Lelli-Chiesa's lifting results to show that $\mathrm{Pic}(S)$ cannot have a lift of a $g^{r'}_d$ for $r' \neq r$.

We can now prove that the maximal Brill–Noether loci in genus 10–19, 22, and 23 are as predicted by Conjecture 6.1. The proof in genus 10–13 uses Brill–Noether theory for curves of fixed gonality and various results distinguishing lattices above. In genus 14–19, 22 and 23, the main strategy to distinguish the expected maximal Brill–Noether loci, for example to show that $\mathcal{M}^r_{g,d} \nsubseteq \mathcal{M}^{r'}_{g,d'}$, is to prove that for a very general K3 surface $(S, H) \in \mathcal{K}^r_{g,d}$, a curve $C \in |H|$ has a g^r_d but not a $g^{r'}_{d'}$. This is done by, first, applying Lemma 6.28 to deduce that C has a g^r_d, and second, assuming that C has a $g^{r'}_{d'}$ and then using various lifting results to produce a line bundle M on S that is numerically incompatible with $\mathrm{Pic}(S)$.

For the rest of the section, we summarize the various arguments, organized by genus.

In low genus, where there are no non-computing linear systems, we argue by the Clifford index of C and can assume that a g^r_d computes the Clifford index of $C \in |H|$. Then Lelli-Chiesa's lifting results [57] suffice to verify Conjecture 6.1 in genus 10–13.

Proposition 6.30. *For any $10 \leq g \leq 13$ and any positive integers r, d, r', d' such that*

- $r' \geq 2$,
- $\rho(g, r, d), \rho(g, r', d') < 0$,
- $\Delta(g, r, d), \Delta(g, r', d') < 0$, *and*
- $2 < \gamma(r', d') \leq \gamma(r, d) \leq \lfloor \frac{g-1}{2} \rfloor$,

there is a polarized K3 surface $(S, H) \in \mathcal{K}^r_{g,d}$ such that a curve $C \in |H|$ admits a g^r_d but not a $g^{r'}_{d'}$. Thus $\mathcal{M}^r_{g,d} \nsubseteq \mathcal{M}^{r'}_{g,d'}$.

Proof. First assume that $r' \geq 2$. We let $(S, H) \in \mathcal{K}^r_{g,d}$ be a very general and $C \in |H|$ a smooth irreducible curve of genus g. As in Proposition 6.10 (i), no lattices obtained by lifting special linear systems on C can be contained in $\mathrm{Pic}(S)$. By Lemma 6.28 (v) we see that $L \otimes \mathcal{O}_C$ is a g^r_d and $\gamma(C) = \gamma(r, d)$. We suppose for contradiction that

C admits a $g^{r'}_{d'}$. We cannot have $\gamma(r', d') < \gamma(r, d)$, as then the g^r_d does not compute the Clifford index of C. Hence $\gamma(r', d') = \gamma(r, d)$. But now [57, Theorem 4.2] shows that we have a Donagi–Morrison lift $M \in \mathrm{Pic}(S)$ of the $g^{r'}_{d'}$, and by Proposition 6.10 (i) again, we see that $\langle H, M\rangle \nsubseteq \mathrm{Pic}(S)$ unless the Donagi–Morrison lift of the $g^{r'}_{d'}$ is of type g^r_d, which only occurs when $r, r' \geq 2$. In this case, the Lazarsfeld–Mukai bundle $E_{C,g^{r'}_{d'}}$ has a quotient E with $\gamma(E) = 0$, and one checks that none of the cases of Lemma 3.21 can occur (for a detailed computation see Proposition 6.35). Thus C cannot admit a $g^{r'}_{d'}$. □

Remark 6.31. When $r' = 1$, case (a) of Lemma 3.21 can occur, hence we assume $r' \geq 2$. In fact, in genus 11, $\mathcal{M}^1_{11,5} \subseteq \mathcal{M}^2_{11,9}$. However, for expected maximal loci, the codimensions of the expected maximal $\mathcal{M}^1_{g,k}$ and $\mathcal{M}^2_{g,d}$ loci rule out similar containments.

Corollary 6.32. *In genus* 10–13*, Conjecture 6.1 holds.The maximal Brill–Noether loci*

- *in genus* 10 *are* $\mathcal{M}^1_{10,5}$ *and* $\mathcal{M}^2_{10,8}$*;*
- *in genus* 11 *are* $\mathcal{M}^1_{11,6}$ *and* $\mathcal{M}^2_{11,9}$*;*
- *in genus* 12 *are* $\mathcal{M}^1_{12,6}$*,* $\mathcal{M}^2_{12,9}$*, and* $\mathcal{M}^3_{12,11}$*;*
- *in genus* 13 *are* $\mathcal{M}^1_{13,7}$*,* $\mathcal{M}^2_{13,10}$*, and* $\mathcal{M}^3_{13,12}$*.*

Proof. Propositions 6.30 and 6.8 suffice to verify the conjecture in genus 10–13. □

6.3.4. Genus 14–15

The arguments in genus 14 and 15 only require the lifting results of Lelli-Chiesa [57] and the preliminary results above.

Proposition 6.33. *In genus* 14*, the maximal Brill–Noether loci are* $\mathcal{M}^1_{14,7}$*,* $\mathcal{M}^2_{14,11}$*, and* $\mathcal{M}^3_{14,13}$*.*

Proof. These loci are the expected maximal Brill–Noether loci in genus 14, thus it remains to show that there are no containments among them. By Proposition 6.8, $\mathcal{M}^1_{14,7} \nsubseteq \mathcal{M}^2_{14,11}$ and $\mathcal{M}^1_{14,7} \nsubseteq \mathcal{M}^3_{14,13}$. By Lemma 6.28 (*iii*), we see that there are curves which admit a g^2_{11} or a g^3_{13} and have maximal gonality $\lfloor \frac{14+3}{2} \rfloor = 8$, whereby $\mathcal{M}^2_{14,11} \nsubseteq \mathcal{M}^1_{14,7}$ and $\mathcal{M}^3_{14,13} \nsubseteq \mathcal{M}^1_{14,7}$. Since $\rho(14,2,11) = -1$ and $\rho(14,3,13) = -2$, and noting that therefore $\mathcal{M}^2_{14,11}$ has codimension 1 and $\mathcal{M}^3_{14,13}$ has codimension at least 2 in $\mathcal{M}_{14}$, we see that $\mathcal{M}^2_{14,11} \nsubseteq \mathcal{M}^3_{14,13}$. Finally, Lelli-Chiesa's lifting of rank 2 linear systems [56] shows that $\mathcal{M}^3_{14,13} \nsubseteq \mathcal{M}^2_{14,11}$. ☐

The proof in genus 15 follows the same argument as genus 14 above.

Proposition 6.34. *In genus* 15*, the maximal Brill–Noether loci are* $\mathcal{M}^1_{15,7}$*,* $\mathcal{M}^2_{15,11}$*, and* $\mathcal{M}^3_{15,14}$*.*

6.3.5. Genus 16–17

In genus 16 and 17, the proofs are slightly complicated by the fact that one cannot expect to always lift a linear system $A \in \mathrm{Pic}(C)$ to a line bundle on S, but under the Donagi–Morrison conjecture, we can at least find a Donagi–Morrison lift, i.e., a line bundle $N \in \mathrm{Pic}(S)$ such that $|A| \subseteq |N \otimes \mathcal{O}_C|$ with $\gamma(N \otimes \mathcal{O}_C) \leq \gamma(A)$, see Definition 4.7.

Proposition 6.35. *The maximal Brill–Noether loci in genus* 16 *are* $\mathcal{M}^1_{16,8}$*,* $\mathcal{M}^2_{16,12}$*, and* $\mathcal{M}^3_{16,14}$*.*

Proof. As above, it remains to show that there are no containments among these loci. One can check, as in Remark 6.29, that for L in $\Lambda^3_{16,14}$, $L \otimes \mathcal{O}_C$ is in fact a g^3_{14}. We note that there are no (-2)-curves in $\Lambda^2_{15,12}$. Hence Lemma 6.28 applies for $\mathrm{Pic}(S)$ either

$\Lambda^2_{16,12}$ or $\Lambda^3_{16,14}$. Thus $\mathcal{M}^2_{16,12} \nsubseteq \mathcal{M}^1_{16,8}$ and $\mathcal{M}^3_{16,14} \nsubseteq \mathcal{M}^1_{16,8}$. Furthermore, we have $\mathcal{M}^1_{16,8} \nsubseteq \mathcal{M}^2_{16,12}$ and $\mathcal{M}^1_{16,8} \nsubseteq \mathcal{M}^3_{16,14}$ from Proposition 6.8. Since $\rho(16,2,12) = -2$ and $\rho(16,3,14) = -4$, we see that $\mathcal{M}^2_{16,12} \nsubseteq \mathcal{M}^3_{16,14}$. It remains to show that there are curves with a g^3_{14} and no g^2_{12}.

Suppose that $\mathrm{Pic}(S) = \Lambda^3_{16,14}$, and suppose C has a line bundle A of type g^2_{12}. Then by [56, Theorem 1], there is a Donagi–Morrison lift of A. It can easily be checked that if the Donagi–Morrison lift M is not of type g^3_{14}, then M can not be contained in $\mathrm{Pic}(S)$. Thus we can assume that M is of type g^3_{14} and $M^2 = 4$. However, by Lemma 3.24, we see that $\gamma(E_{C,A}/N) = 0$, and each of the cases in Lemma 3.21 cannot hold. In case (c), one appeals to [77, Theorem 5.2] which shows that a curve is hyperelliptic only if there is an irreducible curve $B \subset S$ of genus 1 or 2. However, this would yield $B^2 = 0$ or $B^2 = 2$, both of which are too small. Thus there can be no such M, and thus C cannot admit a g^2_{12}. Thus $\mathcal{M}^3_{16,14} \nsubseteq \mathcal{M}^2_{16,12}$. □

The proof in genus 17 follows the same argument as genus 16 above.

Proposition 6.36. *The maximal Brill–Noether loci in genus* 17 *are* $\mathcal{M}^1_{17,9}$, $\mathcal{M}^2_{17,13}$, *and* $\mathcal{M}^3_{17,15}$.

6.3.6. Genus 18

The proof in genus 18 is slightly complicated by the fact that in showing the non-containment $\mathcal{M}^2_{18,13} \nsubseteq \mathcal{M}^3_{18,16}$, the bound in Theorem 5.15 does not rule out the possibility of a $1 \subset 2 \subset 4$ terminal filtration. The other non-containments are similar to the proofs above. We give a proof of this non-trivial non-containment.

Proposition 6.37. *The maximal Brill–Noether loci in genus* 18 *are* $\mathcal{M}^1_{18,9}$, $\mathcal{M}^2_{18,13}$, *and* $\mathcal{M}^3_{18,16}$.

Proof. The only non-containment requiring additional analysis is $\mathcal{M}^2_{18,13} \nsubseteq \mathcal{M}^3_{18,16}$. The other non-containments follow the arguments above.

In Theorem 5.15, the bound on d to ensure that a Donagi–Morrison lift exists for a g^3_{16} on a general $(S,H) \in \mathcal{K}^2_{18,13}$ is 16, and hence we are not guaranteed to have a Donagi–Morrison lift by using Proposition 4.17. However, the bound is sufficient to show that the LM bundle $E_{C,A}$ associated to the g^3_{16} can only have a terminal filtration of type $1 \subset 4$ or $1 \subset 2 \subset 4$. We argue that the terminal filtration of type $1 \subset 2 \subset 4$ cannot exist.

Suppose $\mathrm{Pic}(S) = \Lambda^2_{18,13}$, and that C has g^3_{16}. Lemma 6.28 shows that C has $\gamma(C) = 8$. Suppose also that $E = E_{C,g^3_{16}}$ has a $1 \subset 2 \subset 4$ terminal filtration, which is $0 \subset N \subset M \subset E$ where N is a line bundle and M has rank 2. We show that this leads to a contradiction. We have $c_1(N).c_1(E/N) \geq \gamma(C) + 2$ by Proposition 5.3. Furthermore, C has general Clifford index by Lemma 6.28. Up to replacing N with its saturation, we can assume E/N is a gLM bundle of type (II), and a computation gives $\gamma(E) = c_1(N).c_1(E/N) + \gamma(E/N) - 2$, thus $\gamma(E/N) \leq 2$.

One can easily check that S has no elliptic curves, hence one of the four cases in Proposition 3.27 occur. In case (i) and (ii), one checks the cases in Lemma 3.21, and finds that none can occur. Thus for a smooth irreducible $D \in |\det(E/N)|$, D is either trigonal, a plane quintic, or a plane sextic, see Remark 3.28. If C is hyperelliptic or trigonal, one finds a Donagi–Morrison lift of the g^1_2 or the g^1_3, which cannot be contained in $\mathrm{Pic}(S)$. Thus we may assume $\gamma(D) = 2$. As the condition $(*)$ from [57, Theorem 4.2] applies, we obtain a Donagi–Morrison lift of the g^2_6, which again cannot be contained in $\mathrm{Pic}(S)$. Thus E cannot have a $1 \subset 2 \subset 4$ filtration.

Therefore E can only have a terminal filtration of type $1 \subset 4$, and Conjecture 4.6 holds for the g^3_{16}. The rest of the argument is now similar to the arguments above. $\square$

6.3.7. Genus 19

Proposition 6.38. *The maximal Brill–Noether loci in genus 19 are $\mathcal{M}^1_{19,10}$, $\mathcal{M}^2_{19,14}$, and $\mathcal{M}^3_{19,17}$.*

Proof. To apply Theorem 5.15, it suffices to note that when $\mathrm{Pic}(S) = \Lambda^2_{19,14}$, then we have $\mu \geq 2$ and hence the Donagi–Morrison conjecture holds for a g^3_{17} on a smooth $C \in |H|$, otherwise the argument is similar to Proposition 6.35. □

We include again the example by Knutsen and Lelli-Chiesa Example 4.20. This time with more context in light of our lifting results.

Remark 6.39. In [57, Appendix A, Remark 12], Knutsen and Lelli-Chiesa construct examples of K3 surfaces S of Picard rank 2 such that a smooth irreducible curve $C \subset S$ has a Brill–Noether special linear system A of rank 3 with $\rho(A) = -1$ whose Lazarsfeld–Mukai bundle $E_{C,A}$ admits no effective sub-line bundle. That is, Proposition 4.17 cannot be used to find a Donagi–Morrison lift of A. Here, we give an explicit example and explain how it relates to our results.

We first recall Knutsen and Lelli-Chiesa's construction. For even integers $a, b \geq 4$ and $d = a + b$, let S be a K3 surface with $\mathrm{Pic}(S) = \Lambda^b_{a,d}$. Suppose that $\mathrm{Pic}(S)$ has no classes of self-intersection -2 or 0. There are infinitely many choices of a and b that satisfy these hypotheses, and such that every element of the linear systems $|H|$ and $|L|$ are reduced and irreducible; these are examples of the so-called *Knutsen K3 surfaces* in [2]. Thus general curves $C_1 \in |H|$ and $C_2 \in |L|$ are smooth of genus a and b, and by Lazarsfeld's theorem [53], are Brill–Noether general, in particular, have generic gonality $k_1 = (a + 2)/2$ and $k_2 = (b + 2)/2$, respectively. Let E_1 and E_2 be the LM bundles associated to gonality pencils $g^1_{k_1}$ on C_1 and a $g^1_{k_2}$ on C_2. As these pencils are Brill–Noether general, the LM bundles E_1 and E_2 are simple, hence admit no injective map from an effective line bundle N. A calculation using Remark 3.16 shows that the vector bundle $E = E_1 \oplus E_2$ is a LM bundle associated to a linear system A of type $g^3_{k_1+k_2+d}$ on a smooth irreducible curve $C \in |H + L|$. We note that $g(C) = 2d - 1$, and that $\rho(A) = -1$. However, since E admits no injective map $N \hookrightarrow E$, the linear system A admits no Donagi–Morrison lift, and so Conjecture 4.6

fails for (C, A).

By construction, E has a $2 \subset 4$ terminal filtration. Checking the bound from Lemma 5.4, one finds that $\gamma(C) \leq d-2$, thus C does not have general Clifford index. In fact, one can verify using Lemma 6.28 that $L|_C$ is a line bundle of type g^b_{d+2b-2}, which has $\gamma(L|_C) = d-2$. We note that A is non-computing, and does not compute the special Clifford index $\widetilde{\gamma}(C)$. However, the linear system $L|_C$ does compute $\widetilde{\gamma}(C)$, and has a (Donagi–Morrison) lift by construction. Hence, while this is a counterexample to the Donagi–Morrison conjecture, it does not give a negative answer to Question 6.20.

The first case where such an example shows the failure of Conjecture 4.6 for (C, A) is genus 19, with $a = 6$ and $b = 4$.

The smooth curves in $|H|$ and $|L|$ are Brill–Noether general of genus 6 and 4, so have gonality 4 and 3, respectively. Then we consider a smooth curve $C \in |H + L|$, which has genus 19. The example above gives a Lazarsfeld–Mukai bundle E that is the direct sum of Lazarsfeld–Mukai bundles of gonality pencils on curves in $|H|$ and $|L|$, which is is the Lazarsfeld–Mukai bundle of a g^3_{17} on C. By construction, this LM bundle has a terminal filtration of type $2 \subset 4$ and does not admit any injective map from an effective line bundle, hence the g^3_{17} on C does not admit a Donagi-Morrison lift. However, taking L and restricting it to C, one sees that $L|_C$ is a g^4_{16} which has Clifford index 8, hence C is not Clifford general (as the general Clifford index is 9), and is in fact quite special. The corresponding polarized K3 surface $(S, H + L)$ of genus 19 has $\mathrm{Pic}(S) = \Lambda^4_{19,16}$ with basis $H + L, L$. In the proof of Proposition 6.38, we needed the Donagi–Morrison Conjecture (Conjecture 4.6) for linear systems on curves on a different lattice polarized K3 surface, showing that our bounded version (Theorem 5.15) is in some sense tight (at least in genus 19).

6.3.8. Genus $20 - 21$

We briefly list what is known and summarize the last needed non-containments to verify Conjecture 6.1 in genus 20 and 21.

The expected maximal Brill–Noether loci in genus 20 are $\mathcal{M}^1_{20,10}$, $\mathcal{M}^2_{20,15}$, and $\mathcal{M}^3_{20,17}$, and $\mathcal{M}^4_{20,19}$. We state the following propositions without proof, as they follow the arguments above.

Proposition 6.40. *In genus* 20*, the loci* $\mathcal{M}^1_{20,10}$*,* $\mathcal{M}^2_{20,17}$*, and* $\mathcal{M}^4_{20,19}$ *are maximal. There are also non-containments*

- $\mathcal{M}^3_{20,17} \not\subseteq \mathcal{M}^1_{20,10}$ *and*
- $\mathcal{M}^3_{20,17} \not\subseteq \mathcal{M}^2_{20,17}$.

In fact, the only non-containment that remains to verify Conjecture 6.1 in genus 20 is $\mathcal{M}^3_{20,17} \not\subseteq \mathcal{M}^4_{20,19}$. Current lifting methods do not suffice to prove the last non-containment, as there are no known general lifting results for linear systems of rank 4. If Conjecture 4.6 holds in rank 4, then this would suffice. Another approach to verifying Conjecture 6.1 in genus 20 is to show that the codimension of $\mathcal{M}^3_{20,17}$ is at least the expected value of 4 and the codimension of $\mathcal{M}^4_{20,19}$ is at least the expected value of 5.

Similarly, the expected maximal Brill–Noether loci in genus 21 are $\mathcal{M}^1_{21,11}$, $\mathcal{M}^2_{21,15}$, and $\mathcal{M}^3_{21,18}$, and $\mathcal{M}^4_{21,20}$. And current methods suffice to prove that some expected maximal loci are indeed maximal.

Proposition 6.41. *In genus* 21*, the loci* $\mathcal{M}^1_{21,11}$ *and* $\mathcal{M}^4_{21,20}$ *are maximal. There are also non-containments*

- $\mathcal{M}^2_{21,15} \not\subseteq \mathcal{M}^1_{21,11}$
- $\mathcal{M}^3_{21,18} \not\subseteq \mathcal{M}^1_{21,11}$

- $\mathcal{M}^2_{21,15} \nsubseteq \mathcal{M}^3_{21,18}$
- $\mathcal{M}^3_{21,18} \nsubseteq \mathcal{M}^2_{21,15}$.

Our results reduce the verification of Conjecture 6.1 in genus 21 to the verification of the non-containments $\mathcal{M}^2_{21,15} \nsubseteq \mathcal{M}^4_{21,20}$ and $\mathcal{M}^3_{21,18} \nsubseteq \mathcal{M}^4_{21,20}$. Again, Conjecture 4.6 in rank 4 would suffice. Another approach is by verifying that the codimension of $\mathcal{M}^4_{21,20}$ is the expected value of 4, since $\rho(21, 2, 15) = \rho(21, 3, 18) = -3$ and thus the corresponding loci have codimension 3 in $\mathcal{M}_{21}$.

6.3.9. Genus 22

Proposition 6.42. *The maximal Brill–Noether loci in genus 22 are $\mathcal{M}^1_{22,11}$, $\mathcal{M}^2_{22,16}$, and $\mathcal{M}^3_{22,19}$, and $\mathcal{M}^4_{22,21}$.*

Proof. In genus 22, [12, Corollary 3.5] shows that the loci $\mathcal{M}^2_{22,16}$ and $\mathcal{M}^3_{22,19}$ are distinct. The argument then follows Proposition 6.35. □

6.3.10. Genus 23

Finally, we provide a proof in genus 23. We note that Farkas proved in [26] that the Brill–Noether divisors $\mathcal{M}^1_{23,12}$, $\mathcal{M}^2_{23,17}$, and $\mathcal{M}^3_{23,20}$ are mutually distinct. Our results, and those of Lelli-Chiesa [56], provide a different proof for these non-containments. However, the full proof of Conjecture 6.1 in genus 23 requires our improved lifting results.

Proposition 6.43. *The maximal Brill–Noether loci in genus 23 are $\mathcal{M}^1_{23,12}$, $\mathcal{M}^2_{23,17}$, $\mathcal{M}^3_{23,20}$, and $\mathcal{M}^4_{23,22}$.*

Proof. Since $\rho(23, 1, 12) = \rho(23, 2, 17) = \rho(23, 3, 20) = -1$ and $\rho(23, 4, 22) = -2$, Eisenbud and Harris [24] show that the corresponding loci are irreducible of codimension 1 in $\mathcal{M}_g$ and that $\mathcal{M}^4_{23,22}$ has codimension ≥ 2, hence the other loci cannot be contained in $\mathcal{M}^4_{23,22}$. Since there are no (-2)-curves in the Picard lattices of a general K3 surface in $\mathcal{K}^2_{23,17}$, $\mathcal{K}^3_{23,20}$, and $\mathcal{K}^4_{23,22}$, we see by Lemma 6.28 that none of the loci are contained in $\mathcal{M}^1_{23,12}$. One can check that for a very general K3 surface in $\mathcal{K}^4_{23,22}$, the minimal positive self-intersection is 4. Hence by Theorem 5.15, if $C \in |H|$ had a g^3_{20} then by considering the Donagi–Morrison lifts, one finds that L is the only possible Donagi–Morrison lift of the g^3_{20}. Therefore $\gamma(E/N) = 0$, and one then argues as in the proof of Proposition 6.35. Thus $\mathcal{M}^4_{23,22} \nsubseteq \mathcal{M}^3_{23,20}$. The lifting results in [56] similarly show that $\mathcal{M}^4_{23,22} \nsubseteq \mathcal{M}^2_{23,17}$ and $\mathcal{M}^3_{23,22} \nsubseteq \mathcal{M}^2_{23,17}$. Since the latter two are codimension 1 and irreducible, they are distinct. Thus all of the Brill–Noether loci are distinct. $\square$

Chapter 7

Brill–Noether special K3 surfaces

In this chapter, we present results on the Brill–Noether theory of K3 surfaces informed by Donagi–Morrison type results. Specifically, we answer part of a conjecture of Mukai and Knutsen on the structure of the Picard group of a polarized K3 surface (S, H) if a curve $C \in |H|$ is Brill–Noether special.

Section 7.1

Introduction

In the early 21st century, Mukai introduced a Brill–Noether theory for polarized K3 surfaces, and showed that in genus $g \leq 10$ and $g = 12$, the Brill–Noether general polarized K3 surfaces are precisely those admitting a so-called "Mukai model" in a projective homogeneous variety. While much of this work implicitly circles around the relationship between the Brill–Noether theory of curves on a K3 surface and the Brill–Noether theory of the associated polarized K3 surface, the precise relationship between these two notions is still somewhat mysterious.

Following Mukai [64, Definition 3.8], we say that a polarized K3 surface (S, H) of genus g is Brill–Noether special if there exists a nontrivial line bundle $J \neq H$ such that $h^0(S,J)h^0(S,H-J) \geq h^0(S,H) = g+1$. If both J and $H-J$ are globally generated, this is equivalent to $\rho(g,r,d) = g-(r+1)(g-d+r) < 0$, where $H^2 = 2g-2$, $J^2 = 2r-2$, and $d = J.H$. In this case, we call the sublattice generated by H and J, isomorphic to $\Lambda^r_{g,d}$, a Brill–Noether special marking of $\mathrm{Pic}(S)$.

Example 7.1. Let (S,H) be a K3 surface with $\mathrm{Pic}(S) = \Lambda^1_{g,1}$, that is

$$\mathrm{Pic}(S) = \begin{array}{c|cc} & H & L \\ \hline H & 2g-2 & 1 \\ L & 1 & 0 \end{array}.$$

Note that (S,H) is Brill–Noether special, indeed $\rho(g,1,1) = g-2g < 0$. However, $|H|$ has a base component. We compute that $(H-gL)^2 = -2$, and thus

$$H = gL + ((-2)\text{-curve}),$$

where the (-2)-curve is the fixed component of H. This is the case when S is an elliptic K3 surface with a section L.

Thus we focus on the case when H has no fixed component, that is, when H is ample and basepoint free and hence there are smooth irreducible curves $C \in |H|$.

If a smooth curve $C \in |H|$ is Brill–Noether general, it follows that (S,H) is also Brill–Noether general. To see this, simply restrict a the line bundle J to a C, and note that $J|_C$ has $\rho(C, J|_C) < 0$. We explain the details here.

Proposition 7.2. *If (S,H) is Brill–Noether special, then a smooth curve $C \in |H|$ is Brill–Noether special.*

Proof. Suppose $H = M+N$ with $h^0(M)\cdot h^0(N) \geq g+1$. Consider the long exact sequence for C,

$$0 \to \mathcal{O}_S(-C) \to \mathcal{O}_S \to \mathcal{O}_C \to 0.$$

Tensoring this with M gives

$$0 \to M \otimes \mathcal{O}_S(-C) \to M \to M|_C \to 0.$$

Since $h^0(S, -N) = h^0(S, M \otimes \mathcal{O}_S(-C))$, the long exact sequence in cohomology shows that $h^0(C, M|_C) \geq h^0(S, M)$. Likewise for N. Recall that by adjunciton, $\omega_C = H|_C$. Hence $h^0(C, N|_C) = h^0(C, (H - M)|_C) = h^1(C, M|_C)$. We compute

$$\begin{aligned} h^0(C, M|_C) \cdot h^1(C, M|_C) &= h^0(C, M|_C) \cdot h^0(C, N|_C) \\ &\geq h^0(S, M) \cdot h^0(S, N) \\ &\geq g + 1 > g. \end{aligned}$$

Thus $M|_C$ is Brill–Noether special. Hence C is Brill–Noether special, as desired. □

The converse is an open question, stated by Johnsen and Knutsen [44, Remark 10.2], and attributed to Mukai.

Question 7.3. Let (S, H) be a polarized K3 surface, and $C \in |H|$ a smooth irreducible curve. If C is Brill–Noether special, then is (S, H) is Brill–Noether special?

Lazarsfeld's proof of the Brill–Noether–Petri theorem, Theorem 3.3, in [53] implies that if H generates the Picard group of S, in particular (S, H) is Brill–Noether general, then a smooth curve in the linear system of H is Brill–Noether general. However, the question remains open for K3 surfaces of higher Picard rank.

Theorem 7.4. *Question 7.3 has a positive answer for* $2 \leq g \leq 19$.

Our approach to this question, is to take A and show that a Donagi–Morrison lift M of A gives a Brill–Noether special marking on $\mathrm{Pic}(S)$. The proof would be simplified if we could always find a lift ($M \in \mathrm{Pic}(S)$ such that $M|_C \cong A$), not just a Donagi–Morrison lift of A. However, this is not always the case. The reason we only verify Question 7.3 through genus 19 is that in genus 20 and above, current techniques for lifting line bundles are insufficient. In particular, one would need to lift linear systems of rank 4. In genus 18 and 19, we encounter this problem, but can work around it by finding another linear system with rank 3 which we can lift; this ad hoc solution, however, stops working in higher genus. Throughout the chapter, we let (S, H) be a polarized K3 surface of genus $g \geq 2$, and $C \in |H|$ a smooth irreducible curve of genus g.

Section 7.2

Brill–Noether special K3 surfaces

For convenience, we briefly recall important notions arising in lifting Brill–Noether special linear systems on $C \in |H|$ to a line bundle $L \in \mathrm{Pic}(S)$. We are naturally lead to consider two constraints. Firstly, we have $\rho(g, r, d) < 0$ as the linear system is special. We call the constraint $\rho(g, r, d) \leq 0$ the *Brill-Noether constraint*. Secondly, if the linear system were to lift, then we must have $\operatorname{disc}\mathrm{Pic}(S) < 0$. In analogy with the lifting results of Lelli-Chiesa, we write $L^2 = n = 2r - 2$ and define

$$\Delta(g, r, d) := \operatorname{disc}\left(\Lambda^g_{d,2r-2}\right) = 4(g-1)(r-1) - d^2 = 4(g-1)(r-1) - (\gamma(g, r, d) + 2r)^2,$$

calling the constraint $\Delta \leq 0$ the *Hodge constraint*, as the inequality $\Delta(g, r, d) < 0$ stems from the Hodge index theorem.

Recall that the Noether–Lefschetz divisors $\mathcal{K}^r_{g,d}$ in the moduli space of K3 surfaces of genus g parameterize K3 surfaces with a primitive embedding $\Lambda^r_{g,d} \hookrightarrow \mathrm{Pic}(S)$. For more details, see Section 6.2. In [34], Greer, Li, and Tian study the Picard group of the moduli space of polarized K3 surfaces of genus g, and prove that the non-

Brill–Noether special K3 surfaces are contained a union of certain Noether–Lefschetz divisors. We rephrase the constraints in [34] using the Hodge and Brill–Noether constraint.

Proposition 7.5 ([34, Lemma 2.8])**.** *The locus of non-Brill–Noether general K3 surfaces in $\mathcal{K}_g$ is a union of the Noether–Lefschetz divisors $\mathcal{K}^r_{g,d}$ satisfying $2 \leq d \leq g-1$, $\rho(g,r,d) < 0$, and $\Delta(g,r,d) < 0$.*

Proof. The lower bound in [34, Lemma 2.8] $\sqrt{4(g-1)(r-1)} < d$ is simply $\Delta(g,r,d) < 0$. One of the upper bounds $d \leq g-1$ is trivial. The other upper bound $d \leq r+g-\frac{g+1}{r+1}$ is simply $\rho(g,r,d) < 0$. □

Proposition 7.6. *Let A be a g^r_d on C, and suppose there is a short exact sequence*

$$0 \to N \to E_{C,A} \to E \to 0$$

where N is a line bundle with $h^0(S,N) \geq 2$, and E is a LM bundle with $\gamma(E) = k$. Let $M = \det(E)$. Suppose that $M^2 = 2r'-2$. If $r' > r + \frac{rk}{r-1}$, then E is not stable.

Proof. $E = E_{D,B}$ is a LM bundle for a smooth irreducible curve $D \in |M|$ of genus r' and a line bundle $B \in \mathrm{Pic}(D)$. Since $\gamma(E) = k$ and $\mathrm{rk}(E) = r$, B is a g^{r-1}_{k+2r-2} on D. We compute

$$\rho(g(D), r(B), d(B)) = \rho(r', r-1, k+2r-2),$$

and see that $\rho(g(D), r(B), d(B)) < 0$ if and only if $r' > r + \frac{rk}{r-1}$, hence E is not stable. □

Corollary 7.7. *If the LM bundle E above is stable with $\gamma(E) = 0$ and $\gamma(M|_C) = \gamma(A)$, then $M|_C \cong A$.*

Proof. M is a Donagi–Morrison lift of A. Since E is stable, Proposition 7.6 implies we have $r' = r$, hence as $H.M - M^2 + 2 = \gamma(A) - \gamma(E)$, we have $\deg(M|_C) = \deg(A)$.

Therefore $\gamma(M|_C) = \gamma(A)$ implies that $h^0(C, M|_C) = r + 1$. Thus $M|_C \cong A$, as desired. □

Remark 7.8. If E above is not a LM bundle, then $E^{\vee\vee}$ is a LM bundle of Clifford index $\gamma(E^{\vee\vee}) = \gamma(E) - \ell(\kappa)$, where $\kappa = E^{\vee\vee}/E$ is the 0-dimensional sheaf supported on the points where E is not locally free. Moreover, $E^{\vee\vee}$ is a LM bundle for a $g^{r-1}_{k+2r-2-\ell(\kappa)}$ on a smooth irreducible curve $D \in |M|$. Repeating the same calculation shows that if $r' > r + \frac{r(k-\ell(\kappa))}{r-1}$, then $E^{\vee\vee}$ is not stable.

In any genus, the lifting results of Lelli-Chiesa [57, Theorem 4.2] and Knutsen [46, Lemma 8.3] suffice to verify Question 7.3 when $\gamma(C) \leq \lfloor \frac{g-1}{2} \rfloor$ and $\gamma(A) = \gamma(C)$. In those cases, taking J to be the lift of A provides a Brill–Noether special marking on (S, H). However, in genus ≥ 14, there are non-computing line bundles, which have $\gamma(A) > \gamma(C)$. We verify Question 7.3 in genus $14 - 17$ using Donagi–Morrison lifts.

We begin with a few simple observations to aid the proofs. Throughout this section, let (S, H) be a polarized K3 surface of genus g, $C \in |H|$ a smooth irreducible curve, and A be a g^r_d on C. Suppose also that there is a short exact sequence

$$0 \to N \to E_{C,A} \to E \to 0$$

where N is a line bundle with $h^0(S, N) \geq 2$, and E is a stable gLM bundle with $\gamma(E) = k$. That is, assume Conjecture 4.19 holds for A.

Lemma 7.9. *One has $k = \gamma(E) \leq \gamma(A) - \gamma(C)$.*

Proof. As in the proof of [6, Proposition 3.17], we note that $h^0(C, \det(E)) \geq 2$ and $h^1(C, \det(E)) \geq 2$. Hence $\det(E)|_C$ contributes to the Clifford index of C, and thus $\gamma(\det(E)|_C) \geq \gamma(C)$. Since $\gamma(\det(E)|_C) = \gamma(A) - \gamma(E) - 2h^1(S, N)$, the result follows. □

Proposition 7.10. *If* $\det(E)$ *is of type* $g^{r'}_{d'}$ *and* $\rho(g, r', d') \geq 0$*, then*

$$k = \gamma(E) \leq \gamma(A) + r' - \frac{r'g}{r'+1}.$$

Proof. Suppose that $\det(E)$ is of type $g^{r'}_{\gamma(A)-k+2r'}$. The bound is obtained from

$$\rho(g, r', \gamma(A) - k + 2r') \geq 0.$$

We compute

$$\begin{aligned} &\rho(g, r', \gamma(A) - k + 2r') \geq 0 \\ &\qquad \iff k \leq \frac{g}{r'+1} + r' - g + \gamma(A) \\ &\qquad\qquad = \gamma(A) + r' - \frac{r'g}{r'+1}. \end{aligned}$$

□

Corollary 7.11. *If* $k = \gamma(E) > \gamma(A) + r' - \frac{r'g}{r'+1}$*, then* $\langle H, \det(E)\rangle$ *is a Brill–Noether special marking on* (S, H)*.*

7.2.1. Strong Donagi–Morrison holds for genus 14-19

In genus ≥ 14, there are non-computing line bundles, and thus the lifting results of Lelli-Chiesa and Knutsen [57, 46] where the line bundle computes $\gamma(C)$, do not suffice. To verify Question 7.3 in genus 14-17, we first verify that Conjecture 4.19 holds for the non-computing line bundles, where we use the lifting results of Lelli-Chiesa when $r = 2$ [56], and our work when $r = 3$, Theorem 5.15.

We list the non-computing line bundles in genus 14-19.

- $g = 14$: g^2_{11}, g^3_{13};
- $g = 15$: g^3_{14};

- $g = 16$: g^2_{12}, g^3_{14};
- $g = 17$: g^2_{13}, g^3_{15};
- $g = 18$: g^2_{13}, g^3_{15}, g^3_{16}, g^4_{17};
- $g = 19$: g^2_{14}, g^3_{16}, g^3_{17}, g^4_{18}.

We first show that Conjecture 4.19 holds for each of these linear systems, and then use the Donagi–Morrison lifts to show that $\mathrm{Pic}(S)$ has a Brill–Noether special marking.

Lemma 7.12. *Let (S, H) be a K3 surface of genus g. Suppose there is a line bundle $L \in \mathrm{Pic}(S)$ of type g^r_d with L and $H - L$ globally generated. If $d \leq r$, then (S, H) is Brill–Noether special.*

Proof. Let $r = \frac{2+L^2}{2} \geq 1$, and $d \leq r$. We compute

$$\rho(g, r, d) = g - (r + 1)(g - d + r) = -rg - (r + 1)(r - d) < 0.$$

Thus the sublattice of $\mathrm{Pic}(S)$ generated by H and L is a Brill–Noether special marking of $\mathrm{Pic}(S)$. $\square$

Thus, to show that (S, H) is Brill-Noether special if $C \in |H|$ has a Brill–Noether special g^3_d, we may assume $\mu \geq 4$ in Theorem 5.15.

Lemma 7.13. *Let (S, H) be a polarized K3 surface of genus $14 \leq g \leq 19$ and $C \in |H|$ a smooth irreducible curve. Suppose A is a non-computing line bundle on C, then at least one of the following holds:*

(a) *(S, H) is Brill–Noether special;*

(b) *there is a line bundle $N \hookrightarrow E_{C,A}$ such that $h^0(S, N) \geq 2$ and $E = E_{C,A}/N$ is stable, that is, Conjecture 4.19 holds; or,*

(c) *A is of type g^4_d, and C has a basepoint free complete linear system of type $g^3_{d'}$ with $d' \leq d-1$. Moreover,* (a) *holds or* (b) *holds for the $g^3_{d'}$.*

Proof. For A of type g^2_d, we are always in case (b), the proof follows from [56].

If A is of type g^3_d, we apply Theorem 5.15 and Lemma 7.12. In the proof of Theorem 5.15, we cannot have $m = 0$, as then $c_1(M_1)^2 = 0$ in Lemma 5.6, and M_1 is a gLM of type (II) and if $c_1(M_1)^2 = 0$, then $c_2(M_1) = 0$, which is not the case as M_1 is stable and thus has $c_2(M_1) \geq 2$, as in Remark 3.26. Likewise, by Lemma 7.12, we see that either (S, H) is Brill–Noether special and we are in case (a) or else μ in Theorem 5.15 may be assumed large and we are in case (b).

If A is of type g^4_d as in genus 18 or 19, we show that we are in case (b) or (c). In genus 18, if A is of type g^4_{17}, then C has a g^3_{16}, see [27, 55]. If the g^3_{16} is not basepoint free, then subtracting basepoints yields the result. Likewise in genus 19, if A is of type g^4_{18}, then C has a basepoint free $g^3_{d'}$ with $d' \leq 17$. Applying the above argument to the $g^3_{d'}$ yields the last statement of case (c). □

Theorem 7.14. *Let (S, H) be a polarized K3 surface of genus $14 \leq g \leq 19$ and $C \in |H|$ a smooth irreducible curve. Then C is Brill–Noether special if and only if (S, H) is Brill–Noether special.*

Proof. One direction is Proposition 7.2.

Conversely, suppose that C is Brill–Noether special and has a line bundle A with $\rho(C, A) < 0$. We argue by the Clifford index of C.

If $\gamma(C) < \lfloor \frac{g-1}{2} \rfloor$, the lifting results of Lelli-Chiesa Theorem 4.10 and Knutsen [46, Lemma 8.3] suffice to verify Question 7.3.

Similarly, if $\gamma(C) = \lfloor \frac{g-1}{2} \rfloor$ and $\gamma(A) = \gamma(C)$, then Theorem 4.10 and [46, Lemma 8.3] suffice.

Now suppose that $\gamma(C) = \lfloor \frac{g-1}{2} \rfloor$, and $\gamma(A) > \gamma(C)$, that is, A is non-computing. Now Lemma 7.9, Corollary 7.11, and Lemma 7.13 show that either (S, H) is Brill–

Noether special or the Donagi–Morrison lifts of A give a Brill–Noether special marking on (S, H), as desired. $\square$

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