Limit linear series, Theta characteristics, & reducible Brill-Noether loci.

(joint w/ Montserrat Teixidor i Bigas)

Interested in curves $C \subset \mathbb{P}^r$,

non-degenerate maps $C \to \mathbb{P}^r$ degree d

$\updownarrow$

basepoint free linear series (L, V)

$L \in \mathrm{Pic}^d(C)$ & $V \subseteq H^0(C, L)$, $\dim V = r+1$ $\}$ g^r_d.

Brill-Noether Theorem [Gieseker, Griffiths, Harris, Lazarsfeld '80s]

If $\rho(g,r,d) = g - (r+1)(g-d+r) \geq 0$, then

every curve $C \in M_g$ has a g^r_d ⚠ not nece. basepoint free

if $\rho(g,r,d) < 0$, then general $C \in M_g$ has no g^r_d.

Defn when $\rho(g,r,d) < 0$, the Brill-Noether locus

$$M^r_{g,d} := \{ C \in M_g \text{ admitting a } g^r_d \} \subset M_g.$$

Interested in the geometry of curves in $\mathcal{M}^r_{g,d}$

geometry of $\mathcal{M}^r_{g,d}$: dimension, components, relative positions of $\mathcal{M}^r_{g,d}$

dimension of $\mathcal{M}^r_{g,d}$:

$$\dim \mathcal{M}^r_{g,d} \geq \text{exp dim} = 3g-3+\rho$$

[Pflueger '23, Teixidor; Bigas '23]: when ρ is not too negative, $\mathcal{M}^r_{g,d}$ has a component of exp dim
$(\rho \geq -g+3)$ $(\rho \geq -g+r+1)$

⚠ · But there are examples when $\mathcal{M}^r_{g,d}$ has components of larger dimension too!

Defn $\text{gon}(C) = \min\{k \text{ s.t. } C \text{ admits } g^1_k\}$

$(= \min\{k \text{ s.t. } C \text{ admits deg } k \text{ map to } \mathbb{P}^1\})$

Thm [Jensen-Ranganathan '21]: C general of gon. k. C admits g^r_d iff $\rho_k(g,r,d) \geq 0$.

$$\left(= \max_{\ell \in \{0,\dots,\min\{r,\, g-d+r-1\}\}} \rho(g, r-\ell, d) - \ell k\right)$$

$\rightsquigarrow$ gives inclusions of the form
$\mathcal{M}^1_{g,k} \subseteq \mathcal{M}^r_{g,d}$.

<u>observation:</u> If $\mathcal{M}^1_{g,k} \subseteq \mathcal{M}^r_{g,d}$ s.t.

- $\mathcal{M}^r_{g,d}$ has comp of exp dim
- $\dim \mathcal{M}^1_{g,k} > \text{exp dim } \mathcal{M}^r_{g,d}$

then $\mathcal{M}^r_{g,d}$ has ≥ 2 components.

Can we find more components?

idea: 2 components where the g^r_d's are different

<u>Thm</u> [H-Teixidor i Bigas '25]

- For $r \geq 3$, $g = \binom{r+2}{2}$, $\mathcal{M}^r_{g,g-1}$ has ≥ 2 components
- For $r = 3, 5, 7, \geq 8$, $g = \binom{r+2}{2}$, $\mathcal{M}^r_{g,g-1}$ has ≥ 3 components.

Theta characteristics

<u>Defn</u> A theta characteristic on C is a line bundle s.t. $L^{\otimes 2} = \omega_C$.

called even / odd if $h^0(L)$ is.

$$S_g^r := \{(C,L) \mid C \in M_g,\ L \text{ theta char}, h^0(L) \geq r+1\}$$

$$\begin{tikzcd} S_g^r \arrow[d] \arrow[r] & T_g^r \subseteq M_{g,g-1}^r \\ M_g & \end{tikzcd}$$

$\cong$ $\{C$ w/ L θ-char s.t. $h^0(L) \geq r+1$ (r-dim θ-char.)

[Harris '82]: T_g^r has dimension $\geq 3g-3-\binom{r+1}{2}$
$= \exp\dim T_g^r$

comparing $3g-3+\rho(g,r,g-1)$ & exp dim $Hilb(g,r,g-1)$

<u>Conj</u> [Farkas '05]: For $g \geq \binom{r+2}{2}$, T_g^r has a component of expected dimension

[Farkas '05]: proved for $3 \leq r \leq 11$, $r \neq 10$ with even lower g in some cases.

[Benzo '15] Proved full conj.
used complicated degeneration

[H-Teixidor i Bigas '25]: new proof.

Constructing components of $\mathcal{M}^r_{g,d}$:

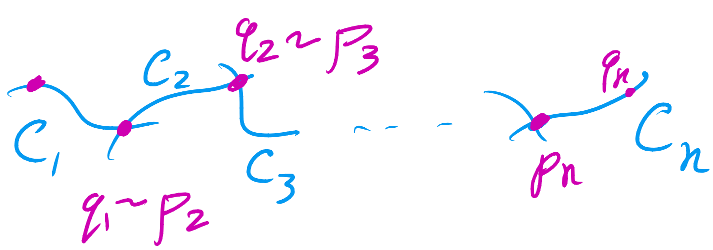
Degenerate to chains of elliptic curves (),

Find a family of expected codimension $(-\rho)$ of chains of ell. curves admitting (limit) g^r_d's.

By the determinantal structure of spaces of linear series, the limit g^r_d's will smooth, and so the family "smooths" to a component of $\mathcal{M}^r_{g,d}$ (really, $\mathcal{M}^r_{g,d}$ has a component of $\geq$ exp dim, but if it were larger, then the family of sing. curves would be larger too)

Limit linear series on chains of elliptic curves.

<u>Defn</u> A <u>limit linear series</u> on a curve



is a collection $(L_i, V_i)_{i=1}^n$ of g^r_d's (L_i, V_i) on C_i

s.t. the vanishing orders

$a^i(p) := 0 \leq a^i_0(p) < \dots < a^i_r(p)$ of sections of V_i

satisfy

$$a^i_j(q_i) + a^{i+1}_{r-j}(p_{i+1}) \geq d.$$

want $|\rho|$ conditions on the chain.

for elliptic curves: forcing $p_i - q_i$ to be torsion, is 1 condition.

<u>Thm</u> [Teixidor i Bigas '23] [Pflueger]

Let $E_1 \cup \cdots \cup E_g$ be a chain of elliptic curves w/ E_i generic except for $E_{i_1}, \ldots, E_{i_e}$ where $l_{i_e}(p_{i_e} - q_{i_e}) \equiv_{lin} 0$.

There is a bijection between limit linear g^r_d's on $E_1 \cup \cdots \cup E_g$ and <u>admissible fillings</u> of $(r+1) \times (g-d+r)$ rectangles w/ $1, \ldots, g$.

1	2	3	6
2	4	5	7
3	6	8	9
5	7	9	10

limit g^3_9 on $E_1 \cup \cdots \cup E_{10}$.

- entries $i_1, \ldots, i_e$ allowed to repeat
- if i_k repeats, l_i divides grid distance
- no entries can be replaced

repeating $-\rho$ indices gives a family of chains of ell. curves w/ limit g^r_d's of codim $-\rho$

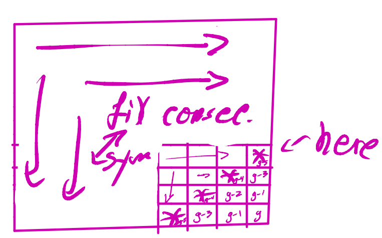
Rk
if $g^r_d \sim$ adm filling, then $\omega_C - g^r_d \sim$ transpose adm filling

so if the adm. filling is symmetric, then $g^r_d = \omega - g^r_d$ so the g^r_d is a theta char!

issue: g^3_9 theta char. on $C \in M_{10}$

1	2	3	5
2	4	6	7
3	6	8	9
5	7	9	10

5's can be replaced by 6's



Rk can always force this issue to be

Fix: smooth $E_5 \cup E_6$ to a genus 2 curve, glued at Weierstrass points, and find aspect explicitly.

1	2	3	4	5	6
	7	8	9	10	11
		12	13	14	16*
			15	17*	18
			17*	19	20
		16*			21

$r=5 \qquad g=\binom{5+2}{2}=21.$

$\rightsquigarrow$ get a component ET_g^r of $\mathcal{M}^r_{g,g-1}$ of curves where $2g^r_d = \omega_C$

<u>Thm</u> [H-Teixidor; Bigas] T^r_g has a comp. of exp. dim for $g \geq \binom{r+2}{2}$.

Can fill non-symmetrically without this issue to get limit g^r_d not a theta char.

$\rightsquigarrow$ get a component N^r_g of $\mathcal{M}^r_{g,g-1}$ of curves where $2g^r_d \neq \omega_C$

<u>Thm</u> [H-Teixidor; Bigas] for $r \geq 3$, $g = \binom{r+2}{2}$ $\mathcal{M}^r_{g,g-1}$ has ≥ 2 components.

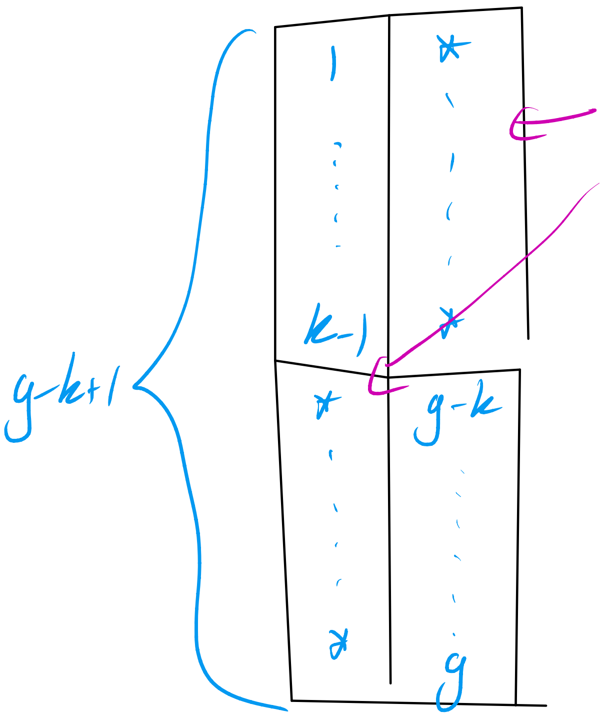
Namely, ET^r_g & N^r_g.

Gonality of curves in ET & N:

<u>Lemma</u> On chain $E_1 \cup \cdots \cup E_g$ a limit of k-gonal curves in $\mathcal{M}^r_{g,d}$, then on at least $-\rho(g,r,d)$ have $m_i(p_i - q_i) = 0$ & $m_i \leq k$.

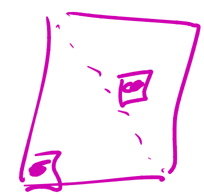
Pf/ limit $g^1_k \rightsquigarrow$ adm filling of $2 \times (g-k+1)$ $\geq 2(g-k+1)-g$ repeats largest possible gaps are:

$\rightarrow \le 2k-2$ non-repeats



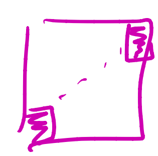
repeated, can only appear once, must have torsion $\le k$. $\blacksquare$

Cor • The general $C \in \mathcal{N}_g^r$ admits no g^1_k for $k \le \begin{cases} r, & r \text{ odd} \\ r-1, & r \text{ even} \end{cases}$



distance $= 2\left\lceil \frac{r+1}{2} \right\rceil$

• general $C \in ET_g^r$ admits no g^1_k for $k \le 2r-1$.



distance $= 2r-1$

Prop $\mathcal{M}^3_{10,9}$ has ≥ 3 components

Pf/ $\mathcal{M}^1_{10,3} \subseteq \mathcal{M}^3_{10,9}$ as $\rho_3(10,3,9) \ge 0$.

$\mathcal{M}^1_{10,4} \not\subseteq$ as $\rho_4(10,3,9) \ge 0$.

Note dim $\mathcal{M}^1_{10,3} = 27 + \rho(10,1,3) = 21$

dim ET^3_{10} = dim $\mathcal{N}^3_{10} = 27 + \rho(10,3,9) = 21$

clearly $ET \neq \mathcal{N}$, suffice to distinguish ET & $\mathcal{N}$ from $\mathcal{M}^1_{10,3}$

By lemmas, curves in $\mathcal{N}^r_{10}$ admit no g^1_3

——— ET^r_{10} admit no g^1_5

Hence $\mathcal{M}^3_{10,9}$ has ≥ 3 components

ET^3_{10}, $\mathcal{N}^3_{10}$, a component containing $\mathcal{M}^1_{10,3}$. $\boxtimes$

In general,

<u>Thm</u> [H-T; B] $r = 3,5,7$, $r \geq 8$, $g = \binom{r+2}{2}$

$\mathcal{M}^r_{g,g-1}$ has ≥ 3 components.

Namely ET^r_g, $\mathcal{N}^r_g$, comp. containing $\mathcal{M}^1_{g,\kappa(g,r,g-1)}$

where $\kappa(g,r,d) = \max\{k \mid \mathcal{M}^1_{g,k} \subset \mathcal{M}^r_{g,d}\}$.

In terms of the BN stratification of $\mathcal{M}_g$, it would also be interesting to find

$$K(g,r,d) := \min \{ k \mid \mathcal{M}^r_{g,d} \subseteq \mathcal{M}^1_{g,k} \}$$

The admissible fillings give some bounds on $K(g,r,d)$