Maximal Brill-Noether loci

Parts joint with Asher Auel & Hannah Larson.

↳ classical Brill-Noether theory.

Brill-Noether theory of alg. curves can be understood as "representation theory for curves".

Q Given an abstract curve C, can we represent C as a curve in $\mathbb{P}^r$ of deg. d?

Study linear systems on curves.

C sm. curve.

Defn A g^r_d on C is a pair

$L \in \mathrm{Pic}^d(C)$ w/ $h^0(L) \geq r+1$, and

$V \subseteq H^0(L)$ of rank r.

$\rightsquigarrow$ gives a map $C \longrightarrow \mathbb{P}^r$ of degree d.

Q when does C have a g^r_d?

Recall: Geometric Riemann-Roch

Let $D = p_1 + \cdots + p_d$ divizor on C

$\phi_K : C \to \mathbb{P}^{g-1}$ can. emb.

$\overline{\phi_K(D)} = \operatorname{span}\{\phi_K(p_1), \ldots, \phi_K(p_d)\}$

Thm $\dim |D| = h^0(C,D) - 1 = d - 1 - \dim \overline{\phi_K(D)}$

Thus if D is a g^r_d, then

$\dim \overline{\phi_K(D)} = d - r - 1$.

we want a r-dim space of these.

C has a g^r_d iff $\phi_K(C)$ has an r-dim'l family of $(d-r-1)$-planes that are d-secant.

inside $G(d-r-1, g-1)$, the planes meeting C once has codim $g-d+r-1$.

we want a plane to meet d times,

so want

$$\underbrace{\dim G(d-r-1, g-1) - \overset{d \text{ indep cond.}}{d}(g-d+r-1)}_{\text{space of planes meeting } C\ d \text{ times (indep)}} \geq \overset{\text{dim of space}}{r}$$

[Griffith - Harris][Lazarsfeld]

Brill-Noether theorem A general curve C of genus g admits a g^r_d iff

$$\rho(g,r,d) = g-(r+1)(g-d+r) \geq 0.$$

We have more precise results as well:

$$G^r_d(C) = \{ g^r_d\text{'s on } C \}$$

$$\begin{aligned} G^r_d(C) &\longrightarrow \operatorname{Pic}^d(C), \\ (A,V) &\longmapsto A \end{aligned}$$

image is called $W^r_d(C)$.

BN Theorem [Gieseker, Griffiths, Harris, Fulton, Kempf, Lazarsfeld]

(i) If $\rho \geq 0$, then $W^r_d(C) \neq \emptyset$ for all $C \in M_g$. ($\Leftrightarrow G^r_d(C) \neq \emptyset$)

(ii) For $C \in M_g$ general, $\dim G^r_d(C) = \rho(g,r,d)$, and it is smooth if >0, then it is irreducible.

E.g./ Not every curve of genus 3 is hyperelliptic.

$$\rho(3,1,2) = 3-(2)(2) = -1.$$

Moreover, there clearly are hyperelliptic curves of every genus:

$C: \ y^2 = f(x)$, with $f(x)$ a degree $2g+2$ poly. w/ distinct roots

(take a 2:1 map $C \to \mathbb{P}^1$ ramified at $2g+2$ points).

<u>Defn</u> Curves admitting a g^r_d with $\rho < 0$ are called <u>Brill-Noether special</u>.

- What are some other BN special curves?

<u>Defn</u> The <u>gonality</u> of a curve is

$$\operatorname{gon}(C) = \min\{k \mid C \text{ admits a } g^1_k\}.$$

By the BN thm, $\operatorname{gon}(C) \leq \lfloor \frac{g+3}{2} \rfloor$, with equality for general C.

Let $\mathcal{M}^1_{g,k} := \{ C \in \mathcal{M}_g \mid \mathrm{gon}(C) \leq k \}$.

we have a stratification of $\mathcal{M}_g$ by gonality:

$$\mathcal{M}^1_{g,2} \subseteq \mathcal{M}^1_{g,3} \subseteq \cdots \subseteq \mathcal{M}^1_{g,\lfloor \frac{g+1}{2} \rfloor} \subseteq \mathcal{M}_g.$$

$\longleftarrow$ more special

$\mathcal{M}^1_{g,k}$ is an irred. var. of codim $-\rho(g,1,d)$.

More generally, we can consider other r,d:

<u>Defn</u> The <u>Brill-Noether loci</u> are

$$\mathcal{M}^r_{g,d} := \{ C \in \mathcal{M}_g \text{ admitting a } g^r_d \}$$

when $\rho(g,r,d) < 0$, $\mathcal{M}^r_{g,d} \subseteq \mathcal{M}_g$ is a proper subvariety.

<u>Facts about BN loci</u>

- $\mathcal{M}^r_{g,d}$ can have multiple components of different dimensions.

- Each component has codimension at most $-\rho$, the expected codim. & codim ≥ 3 when $\rho \leq -3$.

- codim $\mathcal{M}^r_{g,d} = -\rho$ for $-3 \leq \rho \leq -1$
- $\mathcal{M}^r_{g,d}$ irred. when $\rho = -1, -2$. (and distinct)
  ↳ BN divisors used in study of Kodaira dimension of $\mathcal{M}_g$ [Eisenbud-Harris, Choi-Kim-Kim]

- when ρ is not too negative: $\mathcal{M}^r_{g,d}$ (& $\mathcal{G}^r_d$) have components of the exp. dim $-\rho$, and are expected to behave nicely.

↳ <u>Refined BN Theory</u>

<u>Q</u>: What linear systems does a "general" $C \in \mathcal{M}^r_{g,d}$ have?

For fixed gonality:

Thm [Pflueger, Jensen-Ranganathan]

- C general of gonality k, then C has a g^r_d iff

$$\rho_k(g,r,d) = \max_{0 \le \ell \le r'} \rho(g, r-\ell, d) - \ell k \ge 0.$$

$$(r' = \min\{r, g-d+r-1\})$$

Coarser Q: How do BN loci stratify $\mathcal{M}_g$?

Trivial containments:

- $\mathcal{M}^r_{g,d} \subseteq \mathcal{M}^r_{g,d+1}$
- $\mathcal{M}^r_{g,d} \subseteq \mathcal{M}^{r-1}_{g,d-1}$

Q What are the maximal BN loci?

Defn $\mathcal{M}^r_{g,d}$ is <u>expected max'l</u> if $d \le g-1$,

- $\rho(g,r,d) < 0$,
- $\rho(g,r,d+1) \geq 0$, and
- $\rho(g,r-1,d-1) \geq 0$.

$\left(d = r + \left\lceil \frac{gr}{r+1} \right\rceil - 1\right)$.

Rk By the trivial containments, every BN locus (or its Serre dual) is contained

Conj [Auel-H.] For any $g \geq 3$, except 7, 8, 9, the expected max'l BN loci are max'l.

i.e., For each $\mathcal{M}^r_{g,d}$, $\mathcal{M}^s_{g,e}$ exp. max'l BN loci, $\exists$ $C \in \mathcal{M}^r_{g,d}$, $C \notin \mathcal{M}^s_{g,e}$, and $\exists$ $C' \in \mathcal{M}^s_{g,e}$, $C' \notin \mathcal{M}^r_{g,d}$

Known cases on Max BN loci conj:

- if all loci have $\rho = -1$ or all have $\rho = -2$

i.e., $g+1$ or $g+2 \in \{\operatorname{lcm}(1,\dots,n) \mid n \geq 4\}$.

- for $g \leq 23$ [Lelli-Chiesa, Auel-H, Auel-H-Larson, Bud-H]

- many non-containments known [Lelli-Chiesa, Auel-H-Larson, Teixidor i Bigas]

what happens in genus 7, 8, 9?

- secant varieties give non-trivial containments.

E.g.) genus 8

$\mathcal{M}^1_{8,4}$ $\mathcal{M}^2_{8,7}$ are the exp. max'l loci

Let A be a g^1_4, then $\omega_C - A = g^4_{10}$ gives $C \subseteq \mathbb{P}^4$, which will have a 3-secant line, giving a g^2_7. so $\mathcal{M}^1_{8,4} \subseteq \mathcal{M}^2_{8,7}$.

↳ Via gonality stratification. j.w. Asher Auel & Hannah Larson.

Defn

$$\mathcal{K}(g,r,d) = \max\{k \mid \mathcal{M}^1_{g,k} \subseteq \mathcal{M}^r_{g,d}\}$$

Prop If $\mathcal{K}(g,r,d) > \mathcal{K}(g,s,e)$, then $\mathcal{M}^r_{g,d} \not\subseteq \mathcal{M}^s_{g,e}$.

Pf/ Since $\mathcal{K} = \mathcal{K}(g,r,d) > \mathcal{K}(g,s,e)$

so $\mathcal{M}^1_{g,\mathcal{K}} \not\subseteq \mathcal{M}^s_{g,e}$.

$$\begin{array}{ccc} \mathcal{M}^r_{g,d} & \not\subseteq & \mathcal{M}^s_{g,e} \\ \cup\!\mid & \not\nearrow & \\ \mathcal{M}^1_{g,\mathcal{K}} & & \end{array}$$

$\boxtimes$.

E.g./ $\mathcal{K}(8,2,7) = 4$

By BN for curves of fixed gonality,

$$\mathcal{K}(g,r,d) \overset{[P,\, J\text{-}R]}{=} \max\{k \mid \rho_k(g,r,d) \geq 0\}.$$

Prop If $d \le g-1$, $\mathcal{K}(g,r,d) = \begin{cases} \lfloor \frac{d}{r} \rfloor & ; \ g+1 > \lfloor \frac{d}{r} \rfloor + d \\ g+1-d+2r+\lfloor -2\sqrt{-\rho} \rfloor & ; \ \text{else} \end{cases}$

Focus on exp. max'l BN loci:

Thm For $g \geq 9$, $\mathcal{M}^1_{g,\lfloor \frac{g+1}{2} \rfloor} \not\subset \mathcal{M}^r_{g,d}$ $\forall r \geq 2$ exp. max'l.

Pf/ $\mathcal{K}(g,r,d) < \lfloor \frac{g+1}{2} \rfloor$. $\boxtimes$

Lemma: if $\rho(g,r,d) = \rho(g,s,e)$ then $\mathcal{K}(g,r,d) \neq \mathcal{K}(g,s,e)$.

Prop If $r,s \geq 2$, $\rho(g,r,d) = \rho(g,s,e)$ and $\mathcal{M}^r_{g,d}$, $\mathcal{M}^s_{g,e}$ are exp. max'l, then one non-cont. holds

Thm If $\rho(g,r,d) = \rho(g,s,e) = -1$ then $\mathcal{M}^r_{g,d} \not\subset \mathcal{M}^s_{g,e}$.
(and $\mathcal{M}^s_{g,e} \not\subset \mathcal{M}^r_{g,d}$.)

(**Fact**: $\mathcal{M}_{g,d}$ is irred. if $\rho = -1$.)
[Eisenbud, Harris]

<u>Lemma</u> For $\mathcal{M}^s_{g,e}$ exp. max'l,

$$\frac{g}{s+1} + s - 2\sqrt{s+1} < K(g,s,e) \leq \frac{g}{s+1} + s.$$

*Draw sketch of $K(g,r,d)$. *

<u>Thm</u> $\exists\, G(r) \leq 4(r+1)^{5/2} + (r+1)^2 + 2(r+1)^{3/2}$ s.t. $\mathcal{M}^r_{g,d} \not\subset \mathcal{M}^s_{g,e}$ $\forall s > r$, $g \geq G(r)$ exp max'l.

<u>Thm</u> For $g \geq 28$, $\mathcal{M}^2_{g,d}$ exp. max'l is max'l.

Pf/ $\mathcal{M}^2_{g,d} \not\subset \mathcal{M}^s_{g,e}$ $\forall s \geq 3$ by Thm.

RTS $\mathcal{M}^2_{g,d} \not\subset \mathcal{M}^1_{g,\lfloor \frac{g+1}{2} \rfloor}$.

↳ Via K3 surfaces. j.w. Asher Auel.

Strategy: To show $\mathcal{M}^r_{g,d} \not\subseteq \mathcal{M}^s_{g,e}$, find C w/ a g^r_d but no g^s_e on a K3.

① C with a g^r_d:

Let (S,H) be a polarized K3 surface with

$$\mathrm{Pic}(S) = \begin{array}{c|cc} & H & L \\ \hline H & 2g-2 & d \\ L & d & 2r-2. \end{array}$$

Prop $C \in |H|$ sm. irred has gonality $\lfloor \frac{g+3}{2} \rfloor$ and has a g^r_d (might not be $L|_C$, but in nice cases, it is.)

Cor $\mathcal{M}^r_{g,d} \not\subseteq \mathcal{M}^1_{g,\lfloor \frac{g+1}{2} \rfloor}$ for $r \geq 2$ exp max'l.

② what if C has a g^s_e?

Idea: • Then $\exists\, M \in \mathrm{Pic}(S)$ w/ certain numerical properties (*)

• Show such M cannot exist.

(*)

<u>Donagi-Morrison Conj.</u>

If C has a g^s_e w/ $\rho < 0$, then $\exists\, M \in \mathrm{Pic}(S)$ s.t. $g^s_e \subseteq |M|\big|_C$ and M satisfies some numerical properties.

False in general [Lelli-Chiesa-Knutsen]

Bounded versions for $e \leq \underline{\beta(\mathrm{gon}(C), g, \mathrm{Pic}(S))}$

Known: $s=1$ [DM]

$s=2$ [Lelli-Chiesa]

$s=3$ [H].

<u>Proof idea:</u> Study Lazarsfeld-Mukai bundle $\underline{E}$ associated to g^s_e (it is unstable)

<u>Prop</u> If $N \subseteq E$ saturated line bundle w/ $h^0(N) \geq 2$, then $M = \det(E/N)$ works.

To find N:

"Morally"
Consider a destabilizing filtration $0 \subset E_1 \subset E_2 \subset \cdots \subset E_\ell \subset E$ of E s.t. E_{i+1}/E stable, torsion-free, and $\mu(E_i/E_{i-1}) \geq \mu(E_{i+1}/E_i)$.

Show that if $\ell > 1$ & $\operatorname{rk} E_1 > 1$, then $c_2(E) >> 0$, and does not exist on S.

So $\exists\, N \subseteq E$, as desired.

Slogan: $\operatorname{Pic}(S)$ controls which unstable LM bundles exist.