↳ BN loci

- maximal BN loci conj
- fixed gonality Prop

↳ Distinguishing strategy

- Pic(S) lemma
- DM conj (computing γ & rk 2)

↳ rk 3 DM conj

- DM lifts

(↳ Example of distinguishing loci genus 16)

↳ DM conj ⇒ max BN conj

- lattice theory

Maximal Brill-Noether loci via K3 Surfaces

(Joint with Asher Auel)

$\hookrightarrow$ Brill Noether loci

Classical BN theory studies linear systems on curves.

For a general curve C of genus g & a linear system of rank r & degree d, a g^r_d, (map $C \xrightarrow{\deg d} \mathbb{P}^r$)

the BN number $\rho(g,r,d) = g-(r+1)(g-d+r)$ is an important numerical criterion for whether C has a g^r_d:

- if $\rho(g,r,d) \geq 0$, then yes!
- if $\rho(g,r,d) < 0$, then no!

Defn when $\rho(g,r,d)<0$, the BN locus is

$$\mathcal{M}^r_{g,d} = \{C \in \mathcal{M}_g \mid C \text{ has a } g^r_d\}.$$

This is a proper closed subset of $\mathcal{M}_g$.

Eg/ $\mathcal{M}^1_{3,2}$ genus 3 hyperelliptic curves.

$$\rho(3,1,2) = 3-(2)(3-2+1) = 3-4 = -1$$

well-known that not every genus 3 curve is hyperelliptic.

Facts:

- $\mathcal{M}^r_{g,d}$ has expected codim $-\rho$
- when $\rho=-1$, $\mathcal{M}^r_{g,d}$ is irred of codim 1

  ↳ used by Eisenbud, Mumford, Harris Farkas, and Farkas-Jensen-Payne in work on Kodaira dim of $\mathcal{M}_g$

- when $-3 \le \rho \le -1$, $\mathcal{M}^r_{g,d}$ has codim $-\rho$
- $\mathcal{M}^r_{g,d} \subseteq \mathcal{M}^r_{g,d+1}$
- $\mathcal{M}^r_{g,d} \subseteq \mathcal{M}^{r-1}_{g,d-1}$ when $\rho(g,r-1,d-1) < 0$.

Q: How do $\mathcal{M}^r_{g,d}$ stratify $\mathcal{M}_g$?

☆genus 14 drawing☆ $(\gamma(r,d) = d-2r)$

Defn The expected maximal BN loci are the $\mathcal{M}^r_{g,d}$ s.t. for fixed r, d is maximal s.t. $\rho(g,r,d)<0$ & $\rho(g,r-1,d-1)\geq 0$

i.e. maximal wrt containments above.

(every BN locus is contained in an expected maximal BN locus)

Maximal BN locus conj (MBNL)
In genus ≥ 9, the expected maximal BN loci are distinct, and hence are the maximal BN loci.

say That is, given two exp. max. BN loci, $\mathcal{M}^r_{g,d}$ & $\mathcal{M}^{r'}_{g,d'}$, there is a curve with a g^r_d, but not a $g^{r'}_{d'}$.

(point w/ picture)

Rk In genus ≤ 8, this is not true: In genus 8, every BN special curve has a g^2_7, but the g^1_4 locus is also expected maximal. (Mukai)

Rk In genus 23, the exp max are $\mathcal{M}^1_{23,12}$ $\rho=-1$, $\mathcal{M}^2_{23,17}$ $\rho=-1$, $\mathcal{M}^3_{23,20}$ $\rho=-1$, $\mathcal{M}^4_{23,22}$ $\rho=-2$

Farkas proved MBNL conj partially & it's known that codim 1 & codim 2 loci are distinct

Prop For $\rho(g,r,d)\geq 0$, $r(r,d)\geq\left\lfloor\frac{g-1}{2}\right\rfloor+1$, one has $\mathcal{M}^1_{g,\lfloor\frac{g+1}{2}\rfloor}\not\subseteq\mathcal{M}^r_{g,d}$

Pf/ (Pflueger, Jensen-Ranganathan, Larson, ...) $\boxtimes$

Thm (Auel-H) The conjecture holds in genus 9–19, 22–23.

Approach: To distinguish the remaining exp. max BN loci, use linear systems on curves on K3 surfaces to translate the problem to lattice theory of Pic(S). (Farkas, gonality stratification paper)

why K3's: curves on K3's have been constant part of BN theory (Green, Lazarsfeld, DM, Harris)

↳ Curves on K3 surfaces

Let (S,H) be a polarized K3 surface of genus g ($H^2 = 2g-2$)
$C \in |H|$ a sm. irred curve. (C has genus g)

Thm (Lazarsfeld) If $\mathrm{Pic}(S) = \mathbb{Z}H$ then C is BN general

⇝ so need higher Picard rank for C to have BN special systems.

Prop If Pic(S) has a primitive embedding of

$$\Lambda^r_{g,d} = \begin{array}{c|cc} & H & L \\ \hline H & 2g-2 & d \\ L & d & 2r-2 \end{array}, \quad r \geq 2, \quad 0 < d \leq g-1,$$

and L and $H-L$ are basepoint free,

then $L|_C$ is a g^r_d.

What if this curve has a $g^{r'}_{d'}$?

Rk If we had a converse to above prop, we have some lattice cond.
Reid, Saint-Donat, Knutsen, Donagi-Morrison, Lelli-Chiesa have proven various converses.

Idea: If had general converse, "lifting g^r_d"
$\mathcal{A}^r_{g,d}$ & $\mathcal{A}^{r'}_{g,d'}$ are different.

But, we don't have such a converse, ...

Conj (Donagi-Morrison, Lelli-Chiesa)

genus$(C) \geq 2$.

A complete basept free g^r_d on C w/ $d \leq g-1$
and $\rho(g,r,d) < 0$.

Then there is a line bundle M on S
adapted to H s.t. $|A| \subseteq |M|\big|_C$ and
$\gamma(M|_C) \leq \gamma(A) = d-2r$.

Defn Call M a DM lift of A.

Results g^1_2 ✓ (Saint-Donat)

$r=1$, d bound ✓ (Reid)

$r=1$ ✓ (DM)

$r=2$ ✓ (LC)

computes $\gamma(C)$ ✓ (LC) up to finite exceptions

Thm (And-H) DM conj holds for g^3_d if $d <$ bound
depending on $\gamma(C)$ and $\mathrm{Pic}(S)$.

↳ DM conj ⟹ maximal BN loci conj.

If we want to show

$\mathcal{M}^r_{g,d} \not\subseteq \mathcal{M}^{r'}_{g,d'}$ for two exp max. loci,

we suppose $C \subset S$ w/ $\Lambda^r_{g,d} = \mathrm{Pic}(S)$ and

we suppose C has a $g^{r'}_{d'}$.

Then if DM conj holds for $g^{r'}_{d'}$, we get

a line bundle $M \in \mathrm{Pic}(S)$ s.t.

$M^2 = 2s-2$ & $M.H = e$. (M is like a lift of a g^s_e)

s & e depend on r' & d', and there are finitely

many possible such M.

Thus we have $\Lambda^s_{g,e} \subseteq \Lambda^r_{g,d}$.

Lattice cond: such $\Lambda^s_{g,e} \not\subseteq \Lambda^r_{g,d}$.

This lattice condition can be easily checked

Thm (A-H) If this lattice condition holds

and the DM conj holds, for fixed g, then the

max BN locus conj holds.

- Easy to check lattice cond. holds in genus ≤ 88
- DM conj holds in $rk \leq 3$ & suffices to show MBNL conj for genus 9–19, 22, 23

Lazarsfeld-Mukai Bundles

Let A a complete bp-free g^r_d on C.

$$0 \to F_{C,A} \to H^0(C,A) \otimes \mathcal{O}_S \to A \to 0$$

dual

$$0 \to H^0(C,A)^\vee \otimes \mathcal{O}_S \to E_{C,A} \to \omega_C \otimes A^\vee \to 0$$

↳ LM bundle

These were introduced by Lazarsfeld in proof of BN theorem.

Properties of $E_{C,A}$:

$c_1 = [C] = H, \quad c_2 = d, \quad rk = r+1$

$\rho(A) < 0 \implies E_{C,A}$ is not stable

Prop suppose $E_{C,A}$ is unstable and its maximal destabilizing subsheaf is a line bundle N.

Then $M = H - N$ is a DM lift of A

(already implicit in work of Lazarsfeld & DM)

↳ Sketch of proof of rk 3 DM conj

A is a g^3_d, $E_{C,A}$ is rk 4 v.b. on S.

want a filtration like $\mathcal{O} \subset N \subset E_{C,A}$ (1 ⊂ 4 filtration)

so rule out other filtrations

(e.g. $1 \subset 2 \subset 3 \subset 4$)

Thm For a filtration not of type $1 \subset 4$, $c_2(E_{C,A}) \gg 0$.

Idea: $c_2(E_{C,A}) = c_2$ terms + products of c_1's

bound c_2 terms using stability (dim of space of stable sheaves) [Mukai]

bound intersections using slope arguments

Thus when $d <$ bound, the only filtration is $\mathcal{O} \subset N \subset E$ & we have a DM lift.