Refined Brill–Noether Theory

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Abstract

Classical Brill–Noether theory studies linear systems on a general curve in the moduli space $\mathcal{M}_g$ of algebraic curves of genus g. A *refined Brill–Noether theory* studies the linear systems on curves with a given Brill–Noether special linear system, which can be rephrased as understanding the relative positions of (components of) Brill–Noether loci, which parameterize curves with a particular linear series. Using many classical results and recent developments in Brill–Noether theory, we give the relative positions of Brill–Noether loci in genus ≤ 12 and outline expectations in general.

Brill–Noether loci

Given a smooth curve $C \in \mathcal{M}_g$ of genus g, we say a linear series (A, V) is a g^r_d if $A \in \mathrm{Pic}^d(C)$ and $V \subseteq H^0(C, A)$has dimension $r+1$.

The loci

$$\mathcal{M}^r_{g,d} := \{[C] \in \mathcal{M}_g \mid C \text{ admits a linear series of type } g^r_d\} \subseteq \mathcal{M}_g$$

are called *Brill–Noether loci*.

The classical Brill–Noether theorem states that a *general* curve $C \in \mathcal{M}_g$ admits a g^r_d if and only if

$$\rho(g, r, d) = g - (r+1)(g - d + r) \geq 0.$$

In particular, when $\rho(g, r, d) < 0$, $\mathcal{M}^r_{g,d} \subsetneq \mathcal{M}_g$ is a proper subset, parameterizing *Brill–Noether special curves*.

Refined Brill–Noether theory

For a "general" curve $C \in \mathcal{M}^r_{g,d}$, what g^s_e's does it have?

- When $r = 1$, $\mathcal{M}^1_{g,d}$ is irreducible.
  - Refined Brill–Noether theory for curves of fixed gonality (answers question for $r = 1$) [Pflueger, Jensen–Ranganathan, H. Larson, Larson–Larson–Vogt]
- For $r \geq 2$, $\mathcal{M}^r_{g,d}$ can have multiple components of various dimensions! [Pflueger, H.–Teixidor i Bigas]
- Curves in different components can behave very differently!

Relative positions of Brill–Noether loci give a coarse answer.

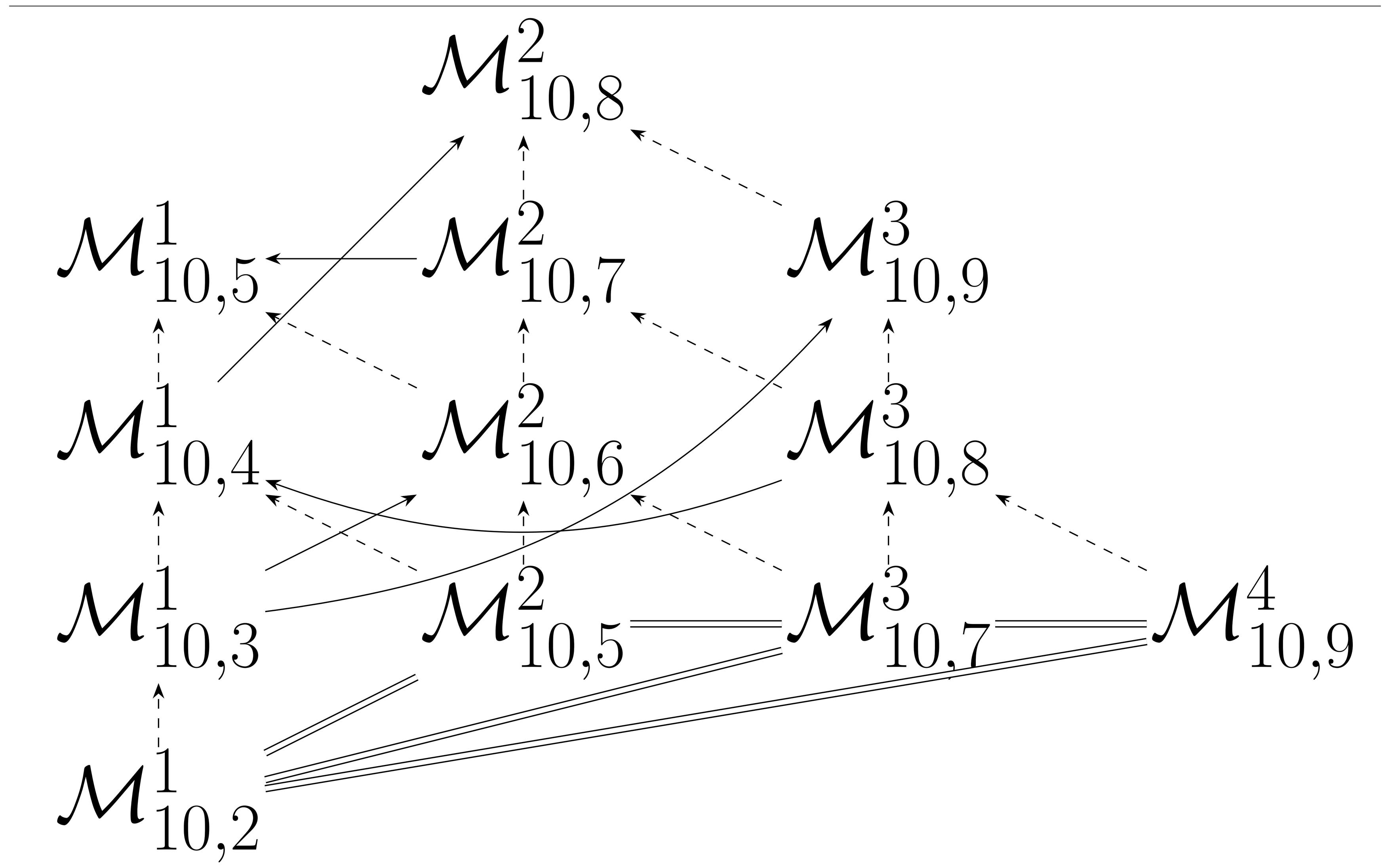
We have trivial containments:

- $\mathcal{M}^r_{g,d} \subset \mathcal{M}^r_{g,d+1}$ by adding a basepoint
- $\mathcal{M}^r_{g,d} \subset \mathcal{M}^{r-1}_{g,d-1}$ by subtracting a non-basepoint

Theorem [H.,'25]. For $g \leq 12$, we identify the relative positions of all Brill–Noether loci.

(Which loci contain $\mathcal{M}^4_{13,12}$??)

Relative positions of Brill–Noether loci in genus 10



Predicting containments

Let $g \geq 3$, $r \geq 1$, and $2 \leq d \leq g-1$, $4(g-1)(r-1) - d^2 < 0$.

Let (S, H) be a polarized K3 surface with $\mathrm{Pic}(S) = \Lambda^r_{g,d}$, where $\Lambda^r_{g,d}$ is the lattice $\mathbb{Z}[H] \oplus \mathbb{Z}[L]$ with intersection matrix

$$\begin{bmatrix} H^2 & H.L \\ H.L & L^2 \end{bmatrix} = \begin{bmatrix} 2g-2 & d \\ d & 2r-2 \end{bmatrix}.$$

For $C \in |H|$ smooth irred., $C \in \mathcal{M}^r_{g,d}$ ($|\mathcal{O}_C(L)|$ is a base point free g^r_d).

Philosophy: Such K3s detect the behavior of the "most general" curves in $\mathcal{M}^r_{g,d}$.

C admits a g^s_e exactly when S admits the Lazarsfeld–Mukai bundle E_{C,g^s_e}.

$$0 \to E_{C,A}{}^{\vee} \to V \otimes \mathcal{O}_S \xrightarrow{ev} A \to 0, \quad \text{where } g^s_e = (A, V)$$

When $\rho(g, s, e) < 0$, E_{C,g^s_e} is unstable, so has some destabilizing filtration

$$0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E_{C,g^s_e}$$

may assume E_{i+1}/E_i stable.

There are many constraints, e.g. the slopes $\mu_{i,j} = \mu_H(E_i/E_{i-j})$ also into a Gelfand–Tsetlin pattern ($\mu_{i,j} \geq \mu_{i+1,j+1} \geq \mu_{i+1,j}$). Since $\mathrm{Pic}(S) = \Lambda^r_{g,d}$ is fixed, an assignment of c_1's to the E_i gives a lower bound on $e = c_2(E_{C,g^s_e})$.

Example $\mathcal{M}^2_{100,d} \overset{?}{\subset} \mathcal{M}^3_{100,e}$

Does a K3 surface S with $\mathrm{Pic}(S) = \Lambda^2_{100,d}$ admit a bundle $E = E_{C,g^3_e}$?

For $d \geq 52$, checking assignments of c_1's, no such bundle exists!

$d = 51$:

There is a destabilizing filtration $E_1 \subset E$ with $\mathrm{rk}\, E_1 = 2$, $c_1(E_1) = H - L$ and $c_2(E_1) = 26$. In fact, $E_{H-L,g^1_{26}} \oplus E_{L,g^1_2}$ is a Lazarsfeld–Mukai bundle of type g^3_{77} ($\mathcal{M}^3_{100,77}$ is maximal).

So we predict a containment $\mathcal{M}^2_{100,51} \overset{?}{\subset} \mathcal{M}^3_{100,77}$.

$d < 50$:

Many destabilizing filtrations appear as d decreases

$38 \leq d \leq 50$: ranks $2 \subset E$, $c_1(E_1) = H - L$;
$d = 37$: ranks $1 \subset E$, $c_1(E_1) = H - 2L$;
$21 \leq d \leq 36$: ranks $1 \subset 2 \subset E$, $c_1(E_1) = H - 2L$, $c_1(E_2) = H - L$;
$d = 20$: Over 2000 filtrations!
$d < 20$: No such K3s.

Conjectures

For sufficiently large g and d, the smallest bounds $c_2(E_n)$ appear to come from destabilizing filtrations of the form $E_1 \subset E_n$ with E_n/E_1 stable, $\mathrm{rk}(E_1) = 2$, and $c_1(E_1) = H - L$.

Conjecture. *For g and d sufficiently large, $d, e \leq g-1$, and $2 \leq r < s$,*

if $e < d - 2r + s + \dfrac{g - d + r + 1}{2} + \dfrac{(s-2)(r-1) - 1}{s-1}$, *then* $\mathcal{M}^r_{g,d} \not\subseteq \mathcal{M}^s_{g,e}$.

Conversely,

if $e \geq d - 2r + s + \dfrac{g - d + r + 1}{2} + \dfrac{(s-2)(r-1) - 1}{s-1}$, *then* $\mathcal{M}^r_{g,d} \subset \mathcal{M}^s_{g,e}$.

Other (non-)containments

There are various *ad-hoc* ways to prove a containment of Brill–Noether loci. For example, using curves on Hirzebruch surfaces [Larson–Vemulapalli], highly secant hyperplanes to curves in $\mathbb{P}^r$, Castelnuovo curves, curves of low Clifford index, and low degree covers of curves using the Castelnuovo–Severi inequality.

Non-containments can also be shown using low degree covers of curves (bi-elliptic curves play a large role in distinguishing loci with $d - 2r = 1$ and $d - 2r \geq 2$). We also use Brill–Noether theory of chains of elliptic curves [Pflueger, Teixidor i Bigas], as well as classical results on Castelnuovo curves and results on the gonality of nodal plane curves.